Uncorrelated and Independent Increments

If the increments $X(t_2)-X(t_1)$ and $X(t_4)-X(t_3)$ of a process $X(t)$ are uncorrelated (or independent) for any $t_1 < t_2 \le t_3 < t_4$, then $X(t)$ is a process with uncorrelated (or independent) increments. The Poisson and the Wiener processes are independent increment processes.

Cross-Correlation and Cross-Covariance

Given two stochastic processes $X(t)$ and $Y(t)$, we define

$$R_{XY}(t_1,t_2) = E\{X(t_1)Y(t_2)\} = R_{YX}(t_2,t_1)$$

as their cross-correlation and

$$C_{XY}(t_1,t_2) = E\{[X(t_1)-\eta_X(t_1)][Y(t_2)-\eta_Y(t_2)]\} = R_{XY}(t_1,t_2) - \eta_X(t_1)\eta_Y(t_2)$$

as their cross-covariance.

Two processes are orthogonal if

$$R_{XY}(t_1,t_2) = 0 \text{ for every } t_1 \text{ and } t_2.$$

They are uncorrelated if

$$C_{XY}(t_1,t_2) = 0 \text{ for every } t_1 \text{ and } t_2.$$

Two processes are independent if the group of random variables $X(t_1),\ldots,\ X(t_n)$ are independent of the group $Y(t_1'),\ldots,\ Y(t_m')$ for any $t_1,\ldots,\ t_n,\ t_1',\ldots,\ t_m'$, i.e.

$$f(x_1,\ldots,x_n,y_1,\ldots,y_m) = f(x_1,\ldots,x_n)f(y_1,\ldots,y_m).$$

Stationary Processes

Definition: Strict-Sense Stationary (SSS)

A random process $X(t)$ is SSS if its statistics are not affected by a shift in the time origin. That is, the two processes $X(t)$ and $X(t+\tau)$ have the same statistics.

A random process $X(t)$ is SSS if its nth order density satisfies the condition

$$f(x_1,\dots,x_n,t_1,\dots,t_n) = f(x_1,\dots,x_n,t_1+\tau,\dots,t_n+\tau) \text{ for any } \tau \text{ and any } n.$$

In particular,

$$f(x;t) = f(x;t+\tau) \quad \text{for any } \tau .$$

This implies that

$$f(x;t) = f(x),$$

that is, the first-order density is independent of time. Similarly, one finds

$$f(x_1,x_2;t_1,t_2) = f(x_1,x_2;t_1-t_2).$$

Hence, it follows that

$$E\{X(t)\} = \eta = const ,$$

$$R(t_1,t_2) = E\{X(t_1)X(t_2)\} = R(t_1-t_2),$$

$$E\{X^2(t)\} = \sigma^2 = const .$$

Definition: Two processes $X(t)$ and $Y(t)$ are jointly SSS if the joint statistics of $X(t)$ and $Y(t)$ are the same as those of $X(t+\tau)$ and $Y(t+\tau)$. This implies that

$$f_{XY}(x,y;t_1,t_2) = f_{XY}(x,y,t_1-t_2),$$

and

$$E\{X(t_1)Y(t_2)\} = R_{XY}(t_1-t_2).$$

Definition: Wide-Sense Stationary (WSS)

A process $X(t)$ is WSS (or weakly stationary) if its mean is constant and its autocorrelation depends only on $\tau = t_1 - t_2$. i.e.,

$$E\{X(t)\} = \eta = const$$

$$E\{X(t+\tau)X(t)\} = R(\tau).$$

Definition: Two processes $X(t)$ and $Y(t)$ are jointly WSS (weakly stationary) if each satisfy the conditions for WSS and their cross-correlation depends only on the time difference. i.e.,

$$E\{X(t+\tau)Y(t)\} = R_{XY}(\tau).$$