ME 529 - Stochastics

Poisson Process

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Poisson Process

Outline

- Poisson Random Variable
- Non-Uniform Case
- Weiner Process
- White Noise Process
- Normal Process

Poisson Process

Poisson Random Variable X(ξ)

$$P\{X(\xi)=k\}=e^{-a}\frac{a^k}{k!}$$

$$f_X(x)=\sum_{k=0}^{\infty}e^{-a}\frac{a^k}{k!}\delta(x-k)$$

$$E\{X\}=a$$

$$E\{X^2\}=a^2+a$$

$$\sigma_x^2=a$$

Poisson Process

Consider probability experiment of placing points at random on a line.
Let $n(t_1, t_2)$ – number of points $\in (t_1,t_2)$;
$X(t) = n(0,t)$ is a Poisson random variable if

i) $P\{n(t_1,t_2)=k\}=e^{-\lambda t}\frac{(\lambda t)^k}{k!}$ $\quad t=t_2-t_1$

ii) If intervals (t_1,t_2) & (t_3,t_4) are non-overlapping, random variables $n(t_1,t_2)$ & $n(t_3,t_4)$ are independent.

Poisson Process

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X(t) is a Poisson process with parameter λt

$$P\{X(t)=k\}=e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

$$E\{X(t)\}=\lambda t$$

$$E\{X^2(t)\}=\lambda^2 t^2+\lambda t$$

Poisson Process-Autocorrelation

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To obtain autocorrelation of X(t), assume $t_2 > t_1$ and consider $E\{X(t_1)[X(t_2)-X(t_1)]\}=E\{X(t_1)X(t_2)\}-E\{X^2(t_1)\}$

Noting that $(0,t_1)$ and (t_1,t_2) do not overlap,

$$E\{X(t_1)\}E\{X(t_2)-X(t_1)\}=R(t_1,t_2)-(\lambda^2 t_1^2+\lambda t_1)$$

or

$$\lambda t_1[\lambda(t_2-t_1)]=R(t_1,t_2)-\lambda^2 t_1^2-\lambda t_1$$

$$R(t_1,t_2)=\lambda^2 t_1 t_2+\lambda t_1 \qquad t_2 \ge t_1$$

Poisson Process-Autocorrelation

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Similarly:

$$R(t_1,t_2)=\lambda^2 t_1 t_2+\lambda t_2 \qquad t_2 \ge t_1$$

$$R(t_1,t_2)=\lambda^2 t_1 t_2+\lambda\min(t_1,t_2)$$

Poisson Process-Non-Uniform

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If the points on the line have non-uniform density λ(t), λt must be replaced by $\int_0^t \lambda(\tau)d\tau$ **:**

$$P\{X(t)=k\}=e^{-\int_0^t \lambda(\tau)d\tau}\frac{\left(\int_0^t \lambda(\tau)d\tau\right)^k}{k!}$$

$$E\{X(t)\}=\int_0^t \lambda(\tau)d\tau$$

$$R(t_1,t_2)=\int_0^{t_1}\lambda(\tau_1)d\tau_1\int_0^{t_2}\lambda(\tau_2)d\tau_2+\int_0^{\min(t_1,t_2)}\lambda(\tau)dz$$

Weiner Process Brownian Motion

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W(t) **is a Weiner Process (Brownian Motion) when:**

i) W(t) **is a normal process with**

ii) $f(w;t) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-\frac{w^2}{2\alpha t}} \qquad F(w;t) = \frac{1}{2} + erf\,\frac{w}{\sqrt{\alpha t}}$

ii) Independent increment process, i.e. $W(t_2) - W(t_1)$ is independent of $W(t_4) - W(t_3)$ if (t_1,t_2) and (t_3,t_4) are non-overlapping

iii) W(0) = 0

Weiner Process Brownian Motion

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Statistics of Weiner Processes

Mean $E\{W(t)\} = 0$

Variance $E\{W^2(t)\} = \alpha t$

Autocorrelation

$$R(t_1,t_2) = E\{W(t_1)W(t_2)\} = \begin{Bmatrix} \alpha t_2 & t_1 \ge t_2 \\ \alpha t_1 & t_2 \ge t_1 \end{Bmatrix} = \alpha \min(t_1,t_2)$$

White Noise Process

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A White Noise Process is the derivative of a Weiner Process. That is,

$$n(t) = \frac{dW(t)}{dt}$$

Mean $E\{n(t)\} = 0$

Autocorrelation $R(t_1,t_2) = \alpha\delta(t_1 - t_2)$

White Noise Process

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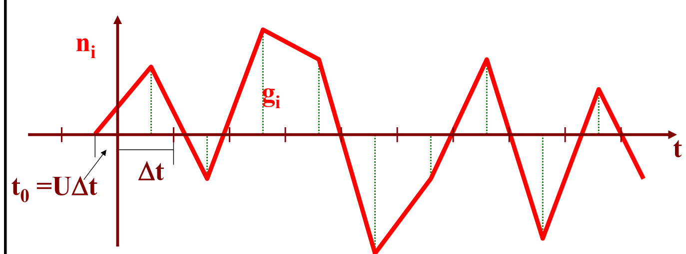

White Noise Process

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Numerical Simulation of White Noise Process

i) For a duration T (~ 20 s) select a small time step Δt (~ 0.01 to 0.05 s) and divide this into $m = T/\Delta t$ (~ 400 to 2000) subintervals

ii) Generate m+1 zero-mean unit-variance normally distributed random numbers $G_1, \ldots, G_{m+1}$. Multiply these by $\left(\frac{2\pi S_0}{\Delta t}\right)^{\frac{1}{2}}$ where S_0 is the constant power spectrum of the white noise. Evaluate $g_i = \left(\frac{2\pi S_0}{\Delta t}\right)^{\frac{1}{2}} G_i$

White Noise Process

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iii) The white noise process is then given by $n(t_0 + i\Delta t) = g_i$, $i = 1,2,\ldots,m+1$, $n(t_0) = 0$; n varies linearly over each subinterval. Here, t_0 is a random variable with uniform density over the subinterval $(-\Delta t, 0)$.

Transformation Form Pair of Uniform Random Variable to Gaussian

$$G_2 = \sqrt{-2\ln U_1}\,\sin 2\pi U_2 \qquad G_1 = \sqrt{-2\ln U_1}\,\cos 2\pi U_2$$

White Noise Process

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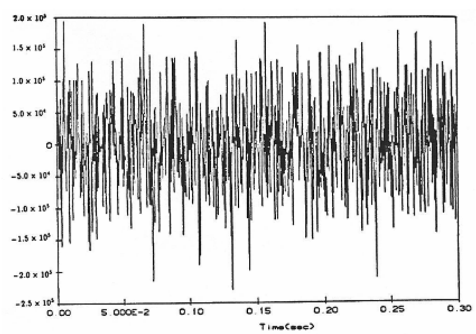
Sample White Noise Process, $d_p = 0.05\ \mu m$



Sample Responses

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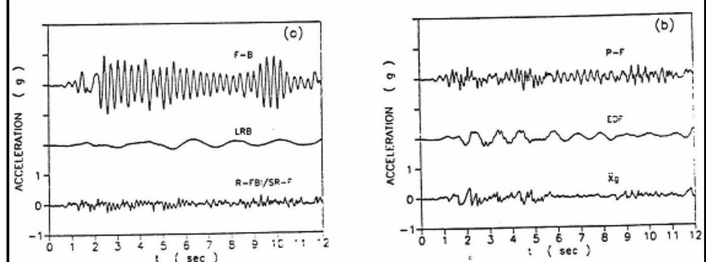
Sample Absolute Acceleration Responses at Top of Structure for El Centro 1940 Quake



Normal Process X(t)

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If $X(t_1), X(t_2), \ldots, X(t_n)$ jointly normal

$$f(x;t) = \frac{1}{\sqrt{2\pi C(t,t)}} e^{\frac{-[x-\eta(t)]^2}{2C(t,t)}}$$

(1st Order Density)

(n^th Order Density)

$$f(x_1,\ldots,x_n;t_1,\ldots,t_n) = \frac{1}{(2\pi)^{\frac{n}{2}}|\Lambda|^{\frac{1}{2}}} exp\left\{-\frac{1}{2}\sum_i\sum_j \Lambda_{ij}^{-1}(x_i-\eta(t_i))(x_j-\eta(t_j))\right\}$$

$\mathbf{\Lambda} = [C(t_i,t_j)]$ $|\Lambda| = \det|\Lambda|$ **(Matrix of Covariance)**

(Note: Linear Combinations of normal processes are also normal)

Concluding Remarks

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- **Poisson Random Variables**
- **Non-Uniform Case**
- **Weiner Process**
- **White Noise Process**
- **Normal Process**

Poisson Process

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Thank you!

Questions?