Stochastic Processes

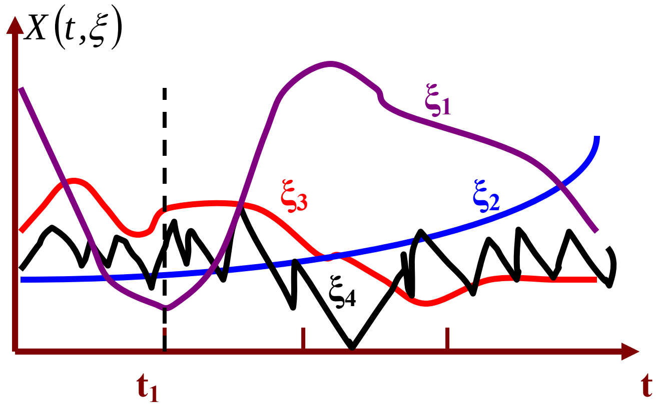
Definition: Given a random experiment

$$\Im : (S, F, P)$$

(with ξ bring the outcomes from space S) to every outcome ξ we assign by a certain rule a time function

$$X(t, \xi).$$

This family of time functions is called a stochastic process. $X(t,\xi)$ is a function of two variables with $t \in (0, T)$ and $\xi \in S$.

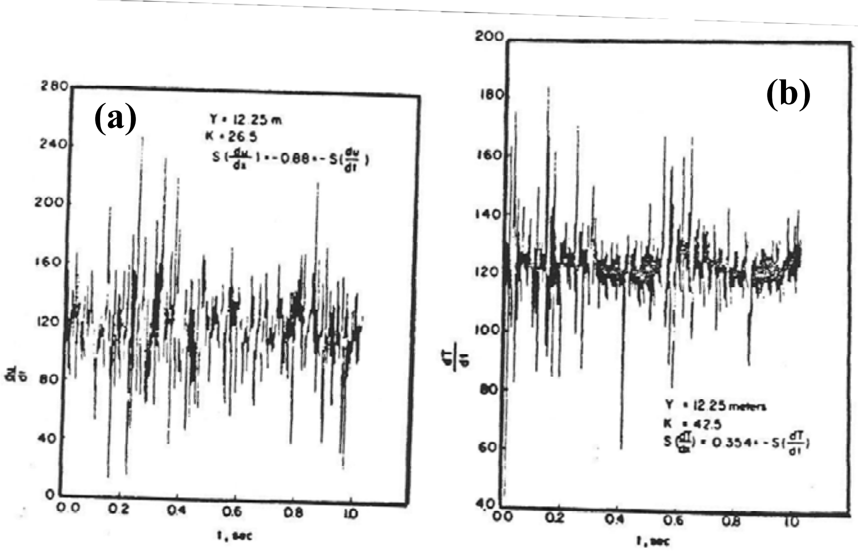


Example of a stochastic process. Sample space has four outcomes.

Meaning of $X(t,\xi)$

- **i)** When t and ξ are variable: A family of time functions.
- **ii)** When t is variable, and ξ is fixed: A single time function.
- **iii)** When t is fixed, and ξ variable: A random variable.
- **iv)** When t and ξ are fixed: A single number.

Clearly, $X_1 = X(t_1, \xi)$, $X_2 = X(t_2, \xi), \ldots,$ $X_n = X(t_n, \xi)$ are n random variables.



Examples of physical stochastic processes. (a)Derivative of air velocity and (b) derivative of air temperature over the ocean.

Alternative Definition:

A stochastic process is a family of random variables $X(t_1)$, $X(t_2)$,… for all t belonging to $(0,T)$. For a discrete parameter stochastic process, this set is finite or countably infinite. For continuous processes this set is non-countably infinite.

Definitions: First-Order Density and Distribution

The first-order distribution function of a stochastic process is defined as

$$F(x,t) = P\{X(t) \le x\}.$$

The first-order density function is defined as

$$f(x;t) = \frac{\partial F(x;t)}{\partial x} \text{ or } f(x;t) = E\{\delta(X(t) - x)\}.$$

Let t_1 and t_2 be two time instances. Consider the random variables $X(t_1)$ and $X(t_2)$.

Definitions: Second-Order Density and Distribution

The second-order distribution function is defined as

$$F(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1 \cap X(t_2) \le x_2\}.$$

The second-order density function is defined as

$$f(x_1, x_2; t_1, t_2) = \frac{\partial^2 F(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}.$$

Definition: Second Order Probablity Density Function (Stratonovich's Definition)

The second order probability density function is defined as

$$f(x_1, x_2; t_1, t_2) = E\{\delta(X(t_1) - x_1)\delta(X(t_2) - x_2)\}.$$

Properties

The second-order distribution and density functions satisfy the usual properties of joint distribution and density functions. For instance

$$F\left(x_1,\infty;t_1,t_2\right)=F\left(x_1;t_1\right),$$

$$f\left(x_1;t_1\right)=\int_{-\infty}^{+\infty}f\left(x_1,x_2;t_1,t_2\right)dx_2\ .$$

Definition: Conditional Density

The conditional density of $X\left(t_1\right)$ given that $X\left(t_2\right)=x_2$ is given as

$$f\left(x_1;t\mid X\left(t_2\right)=x_2\right)=\frac{f\left(x_1,x_2;t_1,t_2\right)}{f\left(x_2,t_2\right)}\ .$$

Definition: Mean Value (Expected Value)

The mean value is defined as

$$\eta(t)=E\{X\left(t,\xi\right)\}=\int_{-\infty}^{+\infty}xf\left(x;t\right)dx\,.$$

Definition: Autocorrelation

The autocorrelation function is defined as

$$R(t_1,t_2)=E\{X\left(t_1\right)X\left(t_2\right)\}=\int_{-\infty}^{+\infty}x_1x_2\,f\left(x_1,x_2;t_1,t_2\right)dx_1dx_2\ .$$

Definition: Autocovariance

The autocovariance function is defined as

$$C(t_1,t_2)=E\{[X\left(t_1\right)-\eta(t_1)][X\left(t_2\right)-\eta(t_2)]\},$$

or

$$C(t_1,t_2)=R(t_1,t_2)-\eta(t_1)\eta(t_2).$$

Definition: Variance

The variance is defined as

$$\sigma_{X(t)}^2(t)=C(t,t)=R(t,t)-\eta^2(t).$$

Examples of Physical Stochastic Processes

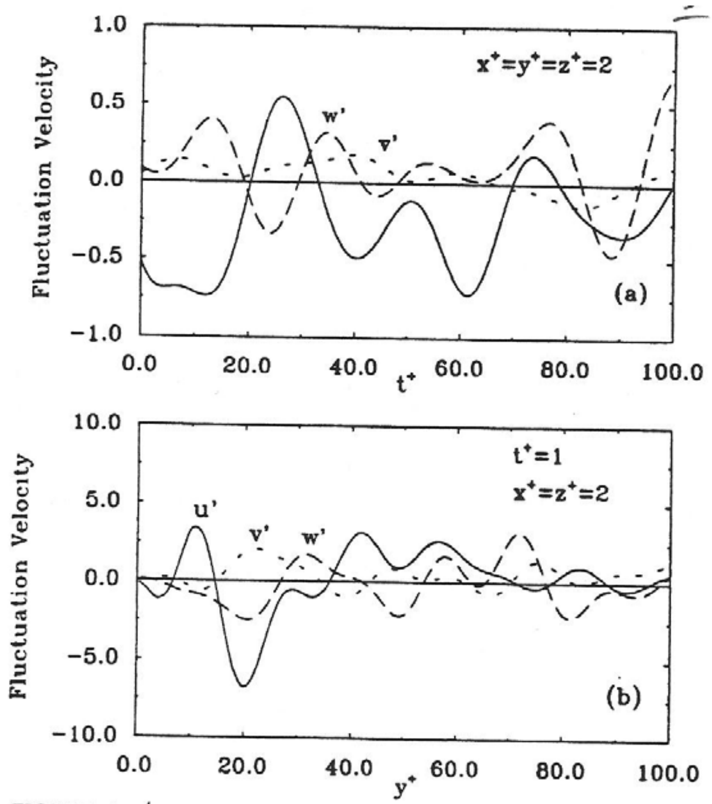


FIGURE 2. Sample time and space variations of fluctuation velocity components.

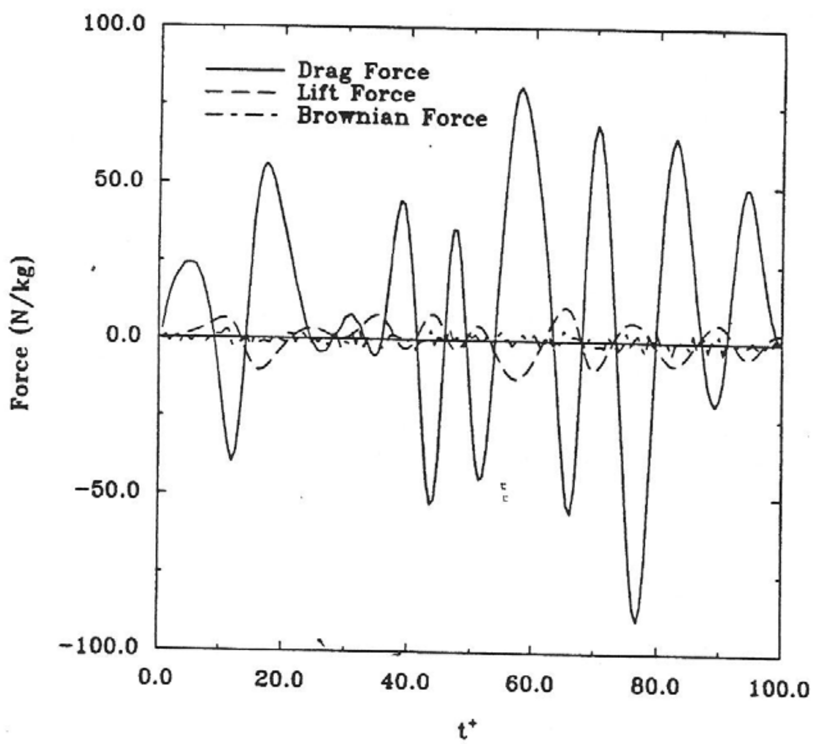


FIGURE 4. The time variations of various forces for a 5-μm particle.

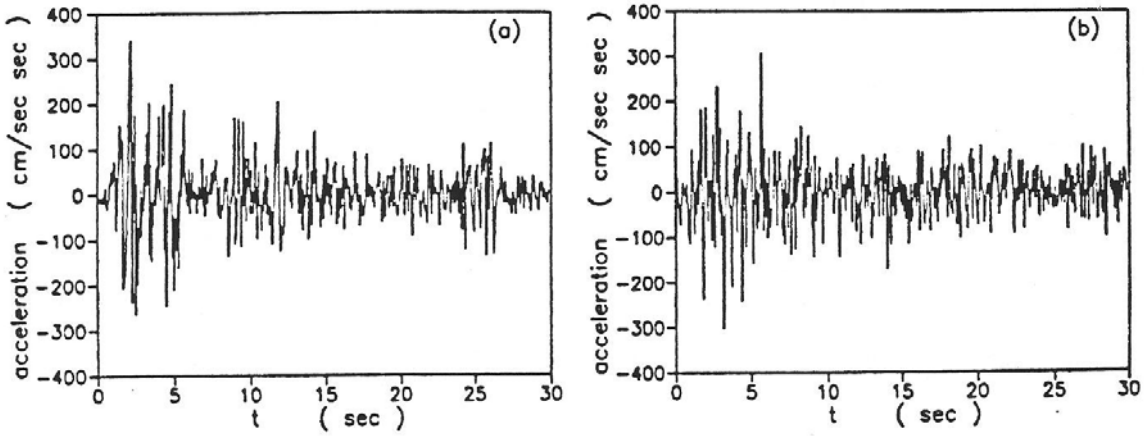


Fig. 3. Accelerograms of the Actual and the Simulated El Centro 1940 Earthquakes

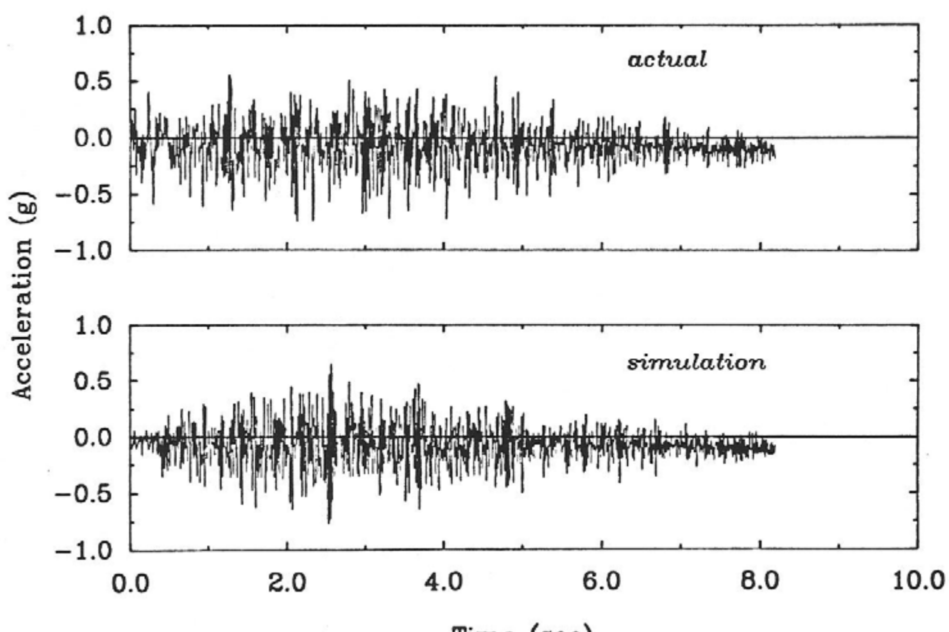


Figure 3: The original and the simulated STS-41 Z lift-off accelerations.