Jointly Normal (Gaussian) Random Variables

Random variables X_1, X_2, … X_n are jointly normal if their joint density $f_{\mathbf{X}}(\mathbf{x}) = f_{X_1,\dots X_n}(x_1,\dots x_n)$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = (2\pi)^{-\frac{n}{2}} |\Lambda|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\eta})^T \cdot \boldsymbol{\Lambda}^{-1} \cdot (\mathbf{x}-\boldsymbol{\eta})\right\},$$

where

$$\eta = E\{\mathbf{x}\}, \; (\eta_j = E\{X_j\}),$$

and

$$\boldsymbol{\Lambda} = \lfloor \mu_{ij} \rfloor \text{ with } \mu_{ij} = E\{(X_i - \eta_i)(X_j - \eta_j)\},$$

is the n x n covariance matrix of $\mathbf{x}$. Here,

$$|\Lambda| = \det|\Lambda| .$$

The jointly normal density function may be rewritten as

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Lambda|^{\frac{1}{2}}} e^{-\frac{1}{2}\sum_i \sum_j \Lambda_{ij}^{-1}(x_i - \eta_i)(x_j - \eta_j)} .$$

The corresponding characteristic function becomes

$$\Phi_{\mathbf{X}}(\mathbf{w}) = e^{i\boldsymbol{\eta}^T \cdot \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\omega}^T \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\omega}},$$

or

$$\Phi_X(\boldsymbol{\omega}) = exp\left\{ i\sum_j \eta_j \omega_j - \frac{1}{2}\sum \Lambda_{k\ell}\omega_k \omega_\ell \right\} .$$

Important Properties of Normal Random Variables:

1. When the first and second order moments (namely $\boldsymbol{\eta}$ and $\boldsymbol{\Lambda}$) are given, the density function is fully specified.
2. If $E\{\mathbf{X}\} = \mathbf{0}$, then the odd moments vanish, i.e. $E\left\{X_1^{k_1} X_2^{k_2} \dots X_n^{k_n}\right\} = 0$ if $\sum_j k_j \sim$ is odd.
3. Event moments are given by $E\{X_1 X_2 \dots X_n\} = \sum_{m_1,\dots,m_n} E\{X_{m1} X_{m2}\} \dots E\left\{X_{m_{n-1}} X_{m_n}\right\}$, and the sum is taken over all possible combinations of $\frac{n}{2}$ pairs of n random variables. (The number of terms in summation is $1 \cdot 3 \cdot 5 \cdot \dots (n-3)(n-1)$, e.g. $E\left\{X_i X_j X_k X_m\right\} = E\left\{X_i X_j\right\} E\{X_k X_m\} + E\{X_i X_k\} E\{X_j X_m\} + E\{X_i X_m\} E\{X_j X_k\}$.
4. Linear combinations of normal random variables are also normal, e.g. if X_i are normal so are $Y_j = \sum_{i=1}^{n} c_{ji} X_i \quad j = 1,2,\dots$

More generally, a linear transformation of normal random variables leads to a set of new normal random variables.

Inequalities

Schwarz Inequality

$$E\{|XY|\} \le \left[E\{X^2\} E\{Y^2\}\right]^{\frac{1}{2}}.$$

Holder Inequality

$$E\{|XY|\} \le \left[E\{|X|^n\}\right]^{\frac{1}{n}} \left[E\{|Y|^m\}\right]^{\frac{1}{m}}, \; n, m > 0, \; \frac{1}{n} + \frac{1}{m} = 1.$$