Central Limit Theorem

Let $X_1,...,X_n$ be a sequence of mutually independent and identically distributed random variables with means η and variances σ^2. Let

$$X = \sum_{j=1}^{n} X_j$$

Define a normalized random variable

$$Y = \frac{X - n\eta}{\sigma n^{1/2}}$$

Then the distribution function of Y converges to a zero mean, unit variance Gaussian distribution as $n \to \infty$.

Proof

$$\Phi_Y(\omega) = E\{e^{i\omega Y}\} = E\{e^{i\Omega(X-n\eta)}\} = E\left\{e^{i\Omega\sum_{j=1}^{n}(X_j-\eta)}\right\} = E\left\{\prod_{j=1}^{n} e^{i\Omega(X_j-\eta)}\right\},$$

where

$$\Omega = \frac{\omega}{\sigma n^{1/2}}.$$

Noting l that $X_1,...,X_n$ are independent random variables, it follows that

$$\begin{aligned}\Phi_Y(\omega) &= \prod_{j=1}^{n} E\{e^{i\Omega(X_j-\eta)}\} = \prod_{j=1}^{n}\left(e^{-i\Omega\eta} E\{e^{i\Omega X_j}\}\right) \\ &= \prod_{j=1}^{n}\left(e^{-i\Omega\eta}\Phi_{Xj}(\Omega)\right) = \left(e^{-i\Omega\eta}\Phi_{Xj}(\Omega)\right)^n\end{aligned}.$$

Note that the characteristic functions are identical. that is.

$$\Phi_{Xj}(\Omega) = \Phi_{X1}(\Omega) = ... = \Phi(\Omega).$$

As $n \to \infty$, $\Omega \to 0$, it follows that

$$\Phi(\Omega) = \sum_k m_k \frac{(i\Omega)^k}{k!},\ m_k = E\{X^k\},$$

and

$$\Phi_Y(\omega) = \left[\left(1 - i\Omega\eta - \frac{\Omega^2\eta^2}{2}...\right)\left(1 + i\Omega\eta - \frac{\sigma^2 + \eta^2}{2}\Omega^2...\right)\right]^n$$
$$= \left[1 + \Omega^2\eta^2 - \frac{\Omega^2\eta^2}{2} - \frac{\sigma^2 + \eta^2}{2}\Omega^2...\right]^n .$$
$$= \left(1 - \frac{\sigma^2\eta^2}{2}...\right)^n$$

Noting

$$\lim_{N\to\infty}\left(1 + \frac{c}{N}\right)^N \to e^c,$$

it then follows that

$$\Phi_Y(\omega) = \left(1 - \frac{\omega^2}{2n}\right)^n \to e^{-\omega^2/2}$$

which is the characteristic function of a zero mean, unit variance normal random variable.