Transformation of Random variables

Function of One Random Variable

Given that $Y(\xi) = g[X(\xi)]$ and the probability distribution of X find the probability distribution of X , we would like to find the probability distribution of Y .

By definition

$$F_Y(y) = P\{Y(\xi) \leq y\} = P\{g(X(\xi)) \leq y\}.$$

Since the statistics of $X(\xi)$ is known, the last term, namely, $P\{g(x) \leq y\}$ could be determined in terms of y . When $F_Y(y)$ is known, then the density function may be found. That is,

$$f_Y(y) = \frac{dF_y(y)}{dy}.$$

Fundamental Transformation Theorem

Given $f_X(x)$ and $Y = g(X)$, then as the probability density function Y is given as

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i(y))}{|g'(x_i(y))|},$$

where $x_i = g^{-1}(y)$ are n real roots for a given y. If for some value of y there is no real root, then

$$f_Y(y) = 0.$$

Justification: By definition,

$$f_Y(y)dy = P\{y < Y \leq y + dy\}$$

Suppose for a given y there are n roots, i.e.

$$y = g(x_i),\ x_i \sim \text{root},\ i = 1,2,...,n$$

Thus

$$f_Y(y)dy = P\{x_1 < X < x_1 + dx_1 \cup ... \cup x_n < X \leq x_n + dx_n\}.$$

or

$$f_Y(y)dy = \sum_{i=1}^{n} f_X(x_i)|dx_i|.$$

Therefore,

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{\frac{dy}{|dx_i|}} = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|}.$$