Characteristic Function

Definition: The characteristic function of a random variable X is defined as

$$\Phi_X(\omega) = E\{e^{i\omega X}\} = \int_{-\infty}^{+\infty} e^{i\omega x} f_X(x)dx .$$

That is, $\Phi(\omega)$ is the Fourier transform of $f(x)$. For discrete random variable's with $f(x) = \sum_j P_j \delta(x - x_j)$, then

$$\Phi(x) = \sum_j P_j e^{i\omega x_j} .$$

Definition: Second characteristic function of a random variable X is defined as

$$\psi(\omega) = \ln \Phi(\omega),$$

or

$$\Phi(\omega) = e^{\psi(\omega)} .$$

Properties of the Characteristic Function

i. $\Phi(\theta) = \int_{-\infty}^{+\infty} f(x)dx = 1$, $\psi(0) = 0$
ii. $|\Phi(\omega)| \leq 1$
iii. $\Phi(\omega)$ is a positive definite function, i.e. $\sum_{m=1}^{n}\sum_{k=1}^{n} \Phi(\omega_m - \omega_k) a_m a_k^* \geq 0$ for any set of complex coefficients a_m. Here a_k^* is the complex conjugate of a_k.

Inversion Formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi(\omega) e^{-i\omega x} d\omega .$$

If $f(x)$ is an even function. i.e., $f(x) = f(-x)$, then $\Phi(\omega)$ is real and even:

$$\Phi(\omega) = \int_{-\infty}^{+\infty} f(x) \cos \omega x d\omega = \frac{1}{\pi}\int_{0}^{\infty} \Phi(\omega) \cos \omega x d\omega .$$

Moment Theorem

Various order moments may be generated from the characteristic function. These are

$$\frac{d^n \Phi(0)}{d\omega^n} = i^n E\{x^n\} = i^n m_n$$

or

$$m_n = E\{x^n\} = \frac{1}{i^n} \frac{d^n \Phi(0)}{d\omega^n}.$$

Using a Taylor series expansion

$$\Phi(\omega) = \sum_{j=0}^{\infty} \frac{m_j}{j!} (i\omega)^j ,$$

the coefficients are related to various moments of random variable.

Moment Generating Function

Definition: The moment generating function of a random variable X is defined as

$$\Phi_X^*(s) = E\{e^{sx}\} = \int_{-\infty}^{+\infty} e^{sx} f_X(x) dx .$$

For discrete random variable's with $f(x) = \sum_j P_j \chi(x - x_j)$, then

$$\Phi^*(s) = \sum_i P_i e^{sx_i} .$$

The moment generating function and the characteristic function of a random variable are related, i.e.

$$\Phi^*(i\omega) = \Phi(\omega), \; \Phi\left(\frac{s}{i}\right) = \Phi^*(s).$$

Moment Theorem

It then follows that

$$E\{x^n\} = \Phi^{*(n)}(0),$$

$$\Phi^*(s) = \sum_{j=0}^{\infty} \frac{m_j}{j!} s^n \ .$$

If $f(x)$ is zero for $x < 0$, then $\Phi^*(s)$ becomes related to the Laplace transform of the density function. i.e.,

$$\Phi(s) = \int_0^{\infty} f(x) e^{sx} dx = L\{f(x)\}|_{s=-s} .$$