# ME 529 - Stochastics

# Characteristic Function

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## Characteristic Function

### Outline

- Characteristic Function
- Inversion Formula
- Moment Theorem
- Moment Generating Function



## Characteristic Function

**Characteristic Function of Random Variable X**

$$\Phi_X(\omega) = E\{e^{i\omega X}\} = \int_{-\infty}^{+\infty} e^{i\omega x} f_X(x)dx$$

**For discrete random variable with**

$$f(x) = \sum_j P_j \delta(x - x_j) \Rightarrow \Phi(x) = \sum_j P_j e^{i\omega x_j}$$

**Second Characteristic Function of X**

$$\psi(\omega) = \ln \Phi(\omega)$$

**or**

$$\Phi(\omega) = e^{\psi(\omega)}$$



## Characteristic Function

**Important Properties**

1. $\Phi(\theta) = \int_{-\infty}^{+\infty} f(x)dx = 1 \qquad \psi(0) = 0$
2. $|\Phi(\omega)| \le 1$
3. $\Phi(\omega)$ **is a positive definite function, i.e.**
   $\sum_{m=1}^{n}\sum_{k=1}^{n} \Phi(\omega_m - \omega_k) a_m a_k^* \ge 0$ **for any set of complex coefficients $a_m$.**

# Characteristic Function

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**Inversion Formula**

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Phi(\omega)e^{-i\omega x}d\omega$$

**If f(x) is an even function Φ(ω) is real and even:**

$$\Phi(\omega)=\int_{-\infty}^{+\infty}f(x)\cos\omega x d\omega=\frac{1}{\pi}\int_{0}^{\infty}\Phi(\omega)\cos\omega x d\omega$$



# Moment Theorem

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**Evaluation of moments from Φ(ω):**

$$\frac{d^n\Phi(0)}{d\omega^n}=i^n E\{x^n\}=i^n m_n$$

**or**

$$m_n = E\{x^n\}=\frac{1}{i^n}\frac{d^n\Phi(0)}{d\omega^n}$$

**Taylor Series Expansion** ⇒

$$\Phi(\omega)=\sum_{j=0}^{\infty}\frac{m_j}{j!}(i\omega)^j$$



# Moment Generating Function

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$$\Phi_X^*(s)=E\{e^{sx}\}=\int_{-\infty}^{+\infty}e^{sx}f_X(x)dx$$

**For discrete random variable with**

$$f(x)=\sum_j P_j\chi(x-x_j)$$ ⇒ $$\Phi^*(s)=\sum_i P_i e^{sx_i}$$

**Note:** $\Phi^*(i\omega)=\Phi(\omega)$ $\Phi\left(\frac{s}{i}\right)=\Phi^*(s)$



# Moment Generating Function

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**Moment Theorem then gives**

$$E\{x^n\}=\Phi^{*(n)}(0)$$ $$\Phi^*(s)=\sum_{j=0}^{\infty}\frac{m_j}{j!}s^n$$

**If f(x) is zero for x < 0:**

$$\Phi(s)=\int_0^{\infty}f(x)e^{sx}dx=L\{f(x)\}|_{s=-s}$$