ME 529 - Stochastics

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Moments of Random Variables- Conditional Distributions and Densities

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Conditional Distributions/Moments

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Outline

- Conditional Distribution and Density
- Expected Value and Moments
- Moments of Normal Random Variable
- Tchevycheff Inequality
- Approximate Evaluation of the Mean and Variance

Conditional Distributions

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Conditional Distribution of X(ξ) given event m

$$F_X(x \mid m) = P\{X(\xi) \le x \mid m\} = \frac{P\{(X \le x) \cap m\}}{P\{m\}}$$

Note that ((X(ξ) ≤ x) ∩ m) is the event consisting of all outcomes ξ such that X(ξ) ≤ x and x ∈ m. The properties of $F_X(x|m)$ are similar to F_X (x).

Conditional Distributions

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Conditional Density of X(ξ) given event m

$$f_X(x \mid m) = \frac{dF_X(x \mid m)}{dx} = \lim_{\Delta x \to 0} \frac{P\{x \le X \le x + \Delta x \mid m\}}{\Delta x}$$

$$f_X(x \mid m) > 0$$

$$\int_{-\infty}^{+\infty} f(x \mid m)\, dx = 1$$

Expected Value

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Expected Value

$$E\{X\} = \int_{-\infty}^{+\infty} x f_X(x)dx = \langle X \rangle$$

For discrete random variable with

$$f_X(x) = \sum_n P_n \delta(x - x_n) \Rightarrow E\{X\} \approx \frac{x_1 + x_2 + \ldots + x_n}{n}$$

Lebesgue Integral in Sample Space

$$E\{X\} = \int_{-\infty}^{+\infty} x f(x)dx = \sum_{i=-\infty}^{+\infty} x_i f(x_i)\Delta x_i = \sum_{i=-\infty}^{+\infty} x_i P\{x_i < X \le x_i + \Delta x_i\} = \int_S X dP$$

Expected Value

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Expected Value of g(X)

$$E\{g(X)\} = \int_{-\infty}^{+\infty} g(x) f_X(x)dx$$

For discrete random variable

$$E\{g(x)\} = \sum_i P_i g(x_i)$$

Expected value is a linear operator:

$$E\left\{\sum_{j=1}^{n} g_j(X)\right\} = \sum_{j=1}^{n} E\{g_j(x)\}$$

Moments of a Random Variable

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Variance $\sigma^2 = E\{x^2\} - \eta^2$

Moments $m_k = E\{x^k\} = \int_{-\infty}^{+\infty} x^k f_X(x)dx$ $m_0 = 1$ $m_1 = \eta$

kth Central Moment $\mu_k = E\{(x-\eta)^k\} = \int_{-\infty}^{+\infty} (x-\eta)^k f_X(x)dx$

$\mu_0 = 1$ $\mu_2 = \sigma^2$ $\mu_1 = 0$ $\mu_3 = m_3 - 3\eta m_2 + 2\eta^3$

Note:

$$\mu_k = E\{(x-\eta)^k\} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \eta^i m_{k-i}$$

Moments of Normal Random Variable

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For a normal random variable

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$$E\{x^n\} = \begin{cases} 1 \cdot 3 \cdot \ldots \cdot (n-1)\sigma^n & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$E\{|X|^n\} = \begin{cases} 1 \cdot 3 \cdot \ldots \cdot (n-1)\sigma^n & n \text{ even} \\ \sqrt{\frac{2}{\pi}} 2^k k! \sigma^{2k+1} & n = 2k+1 \end{cases}$$

Probabilistic Inequality

Tchevycheff Inequality

$$P\{|X-\eta|\ge k\sigma\}\le \frac{1}{k^2}$$

Proof

$$\sigma^2=\int_{-\infty}^{+\infty}(x-\eta)^2 f(x)dx \ge \int_{|x-\eta|\ge k\sigma}(x-\eta)^2 f(x)dx$$

$$\ge k^2\sigma^2\int_{|x-\eta|\ge k\sigma} f(x)dx = k^2\sigma^2 P\{|x-\eta|\ge k\sigma\}$$

$$P\{|X-\eta|\ge k\sigma\}\le \frac{1}{k^2}$$

Approximation

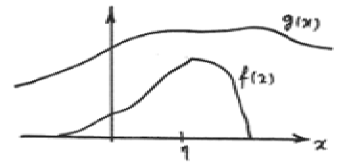
Approximate Evaluation of Mean and Variance of g(X)

$$E\{g(X)\}=\int_{-\infty}^{+\infty} g(x)f(x)dx \approx g(\eta)+g''(\eta)\frac{\sigma^2}{2}$$

$$\sigma^2_{g(x)}\approx g'^2(\eta)\sigma^2$$

$$\eta=E\{X\}$$

$$\sigma^2=E\{(X-\eta)^2\}$$



Moments

Concluding Remarks

- Conditional Density and Distribution
- Expected Value and Moments
- Moments of Normal Random Variable
- Tchevycheff Inequality
- Approximate Evaluation of the Mean and Variance

Moments

Thank you!

Questions?