Extremal Distributions

Let X be the largest of n independent random variables, Y_1, $Y_2,\ldots,Y_n$. Then

$$F_X(x)=P\{Y_i\le x\}=P\{Y_1\le x\}P\{Y_2\le x\}\ldots P\{Y_n\le x\}=F_{Y_1}(x)F_{Y_2}(x)\ldots F_{Y_n}(x). \qquad (1)$$

If Y_i are identically distributed, it follows that

$$F_X(x)=(F_Y(x))^n, \qquad (2)$$

and the corresponding probability density function is given as

$$f_X(x)=n(F_Y(x))^{n-1}f_Y(x). \qquad (3)$$

Gumbel proposed several asymptotic distributions for the extreme values of a random variable. These are described in this section.

i) Type 1: Extremal (Largest) Distribution (Gumbel Distribution)

The distribution of X, the largest of many independent random variables Y (with an exponential upper tail distribution, $F_Y(y)=1-\exp\{-h(y)\}$) is given as

Probability Density

$$f_X(x)=\alpha\exp\{-\alpha(x-u)-e^{-\alpha(x-u)}\} \quad -\infty<x<+\infty. \qquad (4)$$

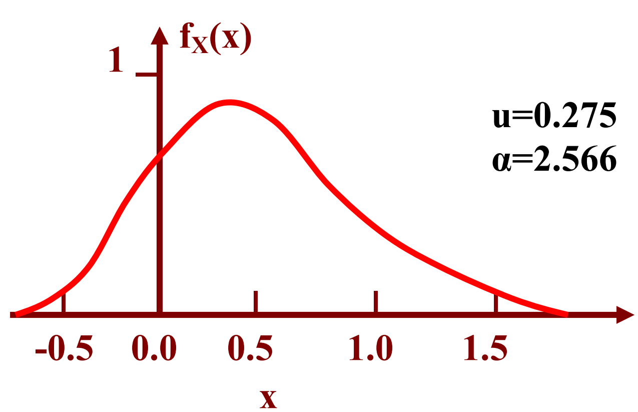
Probability Distribution

$$F_X(x)=\exp\{-e^{-\alpha(x-u)}\}. \qquad (5)$$

Parameters u and α are related to mean and variance of X. These are

$$\mu_X=u+\frac{0.5772}{\alpha}, \qquad (6)$$

$$\sigma_X^2=\frac{\pi^2}{6\alpha^2}. \qquad (7)$$



ii) Type 1: Extremal (Smallest) Distribution (Gumbel Dist.)

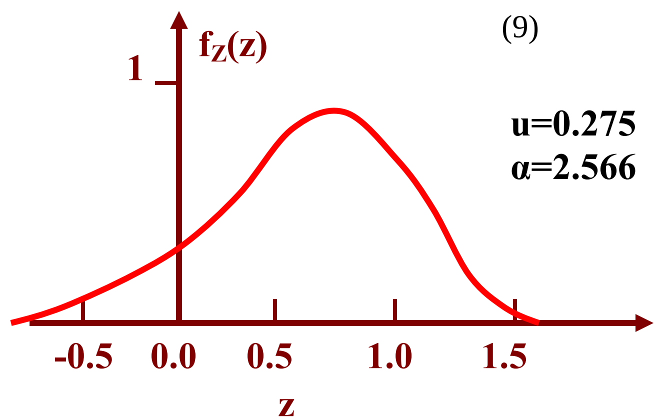
The density and distribution functions of z, the smallest of many independent variables (with an exponential type lower tail distribution) are given as

Probability Density

$$f_Z(z) = \alpha \exp\{\alpha(z-u) - e^{\alpha(z-u)}\} \qquad -\infty < z < +\infty . \tag{8}$$

Probability Distribution

$$F_Z(z) = 1 - \exp\{-e^{\alpha(y-u)}\}, \qquad -\infty < z < +\infty . \tag{9}$$



Here

$$\mu_Z = u - \frac{0.5772}{\alpha}, \tag{10}$$

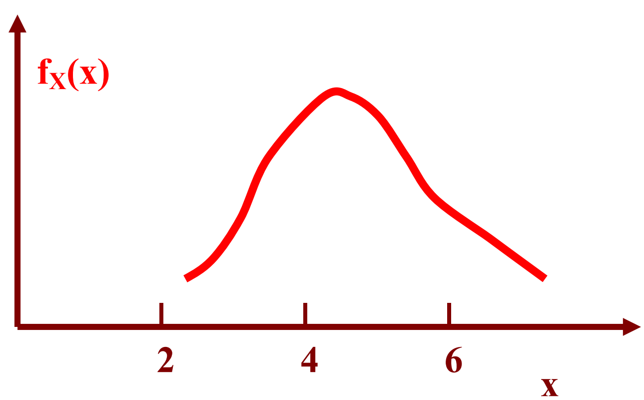
$$\sigma_z^2 = \frac{\pi^2}{6\alpha^2}. \tag{11}$$

iii) Type 2: Extremal (Largest) Distribution (Weibull)

The density and distribution function of X, the largest of many Y_i are given as

Probability Density

$$f_X(x) = \frac{k}{u}\left(\frac{u}{x}\right)^{k+1} e^{-\left(\frac{u}{x}\right)^k}, \quad . x \ge 0 \tag{12}$$



Probability Distribution

$$F_X(x) = e^{-\left(\frac{u}{x}\right)^k}, \qquad x \ge 0 . \tag{13}$$

with

$$\mu_X = u\Gamma\left(1-\frac{1}{k}\right), \qquad k>1, \tag{14}$$

$$\sigma_X^2 = u^2\left[\Gamma\left(1-\frac{2}{k}\right)-\Gamma^2\left(1-\frac{1}{k}\right)\right], \qquad k>2. \tag{15}$$

iv) Type 3: Extremal (Smallest) Distribution

Probability Density

$$f_Z(z) = \frac{k}{u-\ell}\left(\frac{z-\ell}{u-\ell}\right)^{k-1}\exp\left\{-\left(\frac{z-\ell}{u-\ell}\right)^k\right\}, \qquad z\ge\ell. \tag{16}$$

Probability Distribution

$$F_Z(z) = 1-\exp\left\{-\left(\frac{z-\ell}{u-\ell}\right)^k\right\}, \qquad z>\ell, \tag{17}$$

with

$$\mu_Z = \ell+(u-\ell)\Gamma\left(1+\frac{1}{k}\right), \tag{18}$$

$$\sigma_Z^2 = (u-\ell)^2\left[\Gamma\left(1+\frac{2}{k}\right)-\Gamma^2\left(1+\frac{1}{k}\right)\right]. \tag{19}$$

Simulation of a Random Variable with a Known Distribution

We would like to simulate a random variable Y with a known distribution function $F_Y(y)$. Suppose U is a standard uniform random variable with probability density function

$$f_U(u) = \begin{cases} 1 & 0\le u\le 1 \\ 0 & otherwise \end{cases}.$$

It may be shown that

$$Y = F_Y^{-1}(U),$$

has the desired distribution function, $F_Y(y)$.

Examples

i) Exponential

$$f_Y(y) = \lambda e^{-\lambda y} u(y), \qquad F_Y(y) = \left(1 - e^{-\lambda y}\right) u(y),$$

$$Y = -\frac{\ln(1-U)}{\lambda} \text{ or } Y = -\frac{\ln U}{\lambda}.$$

(Note $1-U$ is also uniform)

ii) Weibull

$$f_Y(y) = \alpha\beta y^{\beta-1} e^{-\alpha y^{\beta}} u(y), \qquad F_Y(y) = \left(1 - e^{-\alpha y^{\beta}}\right) u(y),$$

$$Y = \left(-\frac{1}{\alpha}\ln U\right)^{\frac{1}{\beta}}.$$

iii) Gumbel

$$\text{I. } F_Y(y) = \exp\left\{-e^{-\alpha(y-u)}\right\}, \qquad Y = u - \frac{\ln[-\ln U]}{\alpha}.$$

$$\text{II. } F_Y(y) = \exp\left[-\left(\frac{u}{y}\right)^k\right] u(y), \qquad Y = \frac{u}{(-\ln U)^{\frac{1}{k}}}.$$

$$\text{III. } F_Y(y) = 1 - \exp\left(-\left(\frac{y}{u}\right)^k\right) u(y), \qquad Y = u[-\ln(1-U)]^{\frac{1}{k}}.$$

iv) Gaussian

For Gaussian random variable the procedure is different. It may be shown that the pair of random variables defined as,

$$Y_1 = \sqrt{-2\ln U_1}\cos 2\pi U_2,$$

$$Y_2 = \sqrt{-2\ln U_1}\sin 2\pi U_2,$$

are zero mean, unit variance independent Gaussian random variables.

Proof:

For uniform random variables,

$$f_U(u) = \begin{Bmatrix} 1 & 0 < u < 1 \\ 0 & otherwise \end{Bmatrix}, \qquad F_U(u) = \begin{Bmatrix} 1 & u > 1 \\ u & 0 \le u \le 1 \\ 0 & u < 0 \end{Bmatrix}.$$

Consider the transformation

$$Y = F_Y^{-1}(U).$$

Then,

$$F_Y(y) = P(Y \le y) = P\left(F_Y^{-1}(U) \le y\right) = P\{U \le F_Y(y)\} = F_U(F_Y(y)), \qquad 0 < y < 1.$$

Note that for $0 < y < 1$, $F_U(u) = u$. Thus,

$$F_Y(y) = F_U(F_Y(y)) = u(y) = F_Y(y).$$

Alternative using the transformation theorem,

$$Y = F_Y^{-1}(U), \quad f_Y(y) = \sum_i \frac{f_U(u_i(y))}{|g'(u_i(y))|}.$$

Now

$$u = F_Y(y),$$

and

$$du = f_Y(y)dy, \qquad g' = \frac{dy}{du} = \frac{1}{f_Y(y)}.$$

Thus

$$f_Y(y) = \frac{f_u(F_Y(y))}{\frac{1}{f_Y(y)}} = f_u(F_Y(y)) \cdot f_Y(y) = f_Y(y).$$