ME 529 - Stochastics

Clarkson University

Extremal Distributions

Goodarz Ahmadi

Department of Mechanical and Aeronautical Engineering
Clarkson University
Potsdam, NY 13699-5725

Extremal Distributions

Clarkson University

Outline

- Gumbel Asymptotic Distributions
- Type 1: Largest Gumbel Distribution
- Type 1: Smallest Gumbel Distribution
- Type 2: Largest Weibull Distribution
- Type 3: Smallest Distribution
- Simulation of Random Variable With Known Distribution

Extremal Distributions

Clarkson University

Let X be the largest of n independent random variables $Y_1, Y_2, \ldots, Y_n$. Then

$$F_X(x) = P\{\underbrace{Y_i \le x}_{i=1,2,\ldots,n}\} = P\{Y_1 \le x\}P\{Y_2 \le x\}\ldots P\{Y_n \le x\} = F_{Y_1}(x)F_{Y_2}(x)\ldots F_{Y_n}(x)$$

If Y_i are identically distributed:

$$F_X(x) = (F_Y(x))^n \qquad f_X(x) = n(F_Y(x))^{n-1} f_Y(x)$$

Extremal Distributions

Clarkson University

Type 1: (Largest) Gumbel Distribution

Distribution

$$F_X(x) = \exp\{-e^{-\alpha(x-u)}\}$$

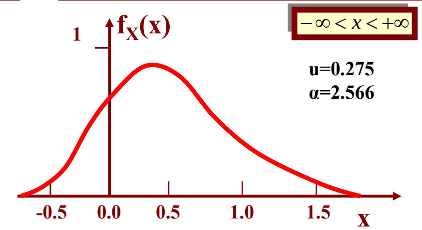
Density

$$f_X(x) = \alpha \exp\{-\alpha(x-u) - e^{-\alpha(x-u)}\}$$

$$-\infty < x < +\infty$$

$$\mu_X = u + \frac{0.5772}{\alpha}$$

$$\sigma_X^2 = \frac{\pi^2}{6\alpha^2}$$



Extremal Distributions

Clarkson University

Type 1: (Smallest) Gumbel Distribution

Distribution

$$F_Z(z)=1-\exp\left\{-e^{\alpha(y-u)}\right\}$$

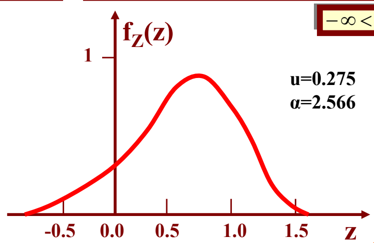
Density

$$f_Z(z)=\alpha\exp\left\{\alpha(z-u)-e^{\alpha(z-u)}\right\}$$

$-\infty<z<+\infty$

$$\mu_Z=u-\frac{0.5772}{\alpha}$$

$$\sigma_Z^2=\frac{\pi^2}{6\alpha^2}$$



Extremal Distributions

Clarkson University

Type 2: (Largest) Weibull Distribution

Distribution

$$F_X(x)=e^{-\left(\frac{u}{x}\right)^k}$$

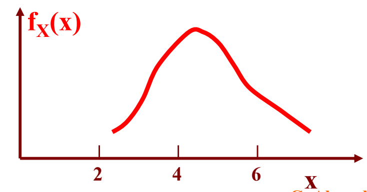
Density

$$f_X(x)=\frac{k}{u}\left(\frac{u}{x}\right)^{k+1}e^{-\left(\frac{u}{x}\right)^k}$$

$x\ge 0$

$$\mu_X=u\Gamma\left(1-\frac{1}{k}\right)$$ $k>1$

$$\sigma_X^2=u^2\left[\Gamma\left(1-\frac{2}{k}\right)-\Gamma^2\left(1-\frac{1}{k}\right)\right]$$ $k>2$



Extremal Distributions

Clarkson University

Type 3: (Smallest) Extremal Distribution

Distribution

$$F_Z(z)=1-\exp\left\{-\left(\frac{z-\ell}{u-\ell}\right)^k\right\}$$

$z>\ell$

Density

$$f_Z(z)=\frac{k}{u-\ell}\left(\frac{z-\ell}{u-\ell}\right)^{k-1}\exp\left\{-\left(\frac{z-\ell}{u-\ell}\right)^k\right\}$$

$z\ge\ell$

$$\mu_Z=\ell+(u-\ell)\Gamma\left(1+\frac{1}{k}\right)$$

$$\sigma_Z^2=(u-\ell)^2\left[\Gamma\left(1+\frac{2}{k}\right)-\Gamma^2\left(1+\frac{1}{k}\right)\right]$$

Simulation of Random Variables

Clarkson University

Simulation of Random Variables with a Known Distribution $F_Y(y)$

Given that U is uniform random variable with ⇒

$$f_U(u)=\begin{cases}1 & 0\le u\le 1\\ 0 & otherwise\end{cases}$$

Then

$$Y=F_Y^{-1}(U)$$

has the desired distribution function $F_Y(y)$.

Examples

Clarkson University

Exponential

$$f_Y(y)=\lambda e^{-\lambda y}u(y)$$

$$F_Y(y)=\left(1-e^{-\lambda y}\right)u(y)$$

$$Y=-\frac{\ln(1-U)}{\lambda}$$ **or** $$Y=-\frac{\ln U}{\lambda}$$

Weibull

$$f_Y(y)=\alpha\beta y^{\beta-1}e^{-\alpha y^{\beta}}u(y)$$

$$F_Y(y)=\left(1-e^{-\alpha y^{\beta}}\right)u(y)$$

$$Y=\left(-\frac{1}{\alpha}\ln U\right)^{\frac{1}{\beta}}$$

ME 529 - Stochastics G. Ahmadi

Examples

Clarkson University

Gumbel

$$F_Y(y)=\exp\{-e^{-\alpha(y-u)}\}$$

$$Y=u-\frac{\ln[-\ln U]}{\alpha}$$

$$F_Y(y)=\exp\left[-\left(\frac{u}{y}\right)^k\right]u(y)$$

$$Y=\frac{u}{(-\ln U)^{\frac{1}{k}}}$$

$$F_Y(y)=1-\exp\left(-\left(\frac{y}{u}\right)^k\right)u(y)$$

$$Y=u[-\ln(1-U)]^{\frac{1}{k}}$$

Gaussian

$$Y_1=\sqrt{-2\ln U_1}\cos 2\pi U_2$$

$$Y_2=\sqrt{-2\ln U_1}\sin 2\pi U_2$$

ME 529 - Stochastics G. Ahmadi

Extremal Distributions

Clarkson University

Concluding Remarks

- Type 1: Smallest Gumbel
- Type 1: Largest Gumbel
- Type 2: Largest Weibull
- Type 3: Smallest
- Simulation of Random Variable With Known Distribution

ME 529 - Stochastics G. Ahmadi

Extremal Distributions

Clarkson University

Thank you!

Questions?

ME 529 - Stochastics G. Ahmadi