Random Variables

Given a random experiment, $\Im:(S,F,P)$, a real valued function $X(\xi)$ defined on the probability space is called a random variable. That is, to every outcome, ξ , a number $X(\xi)$ is assigned by a certain rule. A random variable (RA) $X(\xi)$ is subject to the following two conditions:

i) For every real number x, the set $\{\xi : X(\xi) \le x\}$ is an event in F .

ii) $P(x=\infty)=0$, $P(x=-\infty)=0$.

Probability Distribution Function

Definition: The distribution function of a random variable $X(\xi)$ is a function $F_X(x)$ defined by

$$F_X(x) = P\{X(\xi) \le x\}, \qquad -\infty \le x \le +\infty .$$

Properties of $F_X(x)$

Important properties of the probability distribution function are:

i) $F(-\infty)=0$, $F(+\infty)=1$.

ii) $F(x)$ is a nondecreasing function of x, i.e. $F(x_1) \le F(x_2)$ if $x_1 \le x_2$.

iii) $F(x)$ is continuous from the right. i.e., $F(x^+) = F(x)$, where $F(x^+) = \lim_{\varepsilon \to 0} F(x+\varepsilon)$, $\varepsilon > 0$.

iv) If $F(x_0)=0$, then $F(x)=0$ for every $x \le x_0$.

v) $P\{X(\xi) > x\} = 1 - F(x)$

vi) $P\{x_1 < X \le x_2\} = F(x_2) - F(x_1)$

vii) $P\{X = x\} = F(x) - F(x^-)$, where $F(x^-) = \lim_{\varepsilon \to 0} F(x-\varepsilon)$

viii) $P\{x_1 \le X \le x_2\} = F(x_2) - F(x_1^-)$.

Density Function

The probability density function of a random variable $X(\xi)$ is defined as

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

Continuous and Discrete Random Variable

A random variable $X(\xi)$ is called a continuous random variable if its associated distribution function is continuous and differentiable almost everywhere. For a discrete random variable, the distribution function takes the shape of a staircase with a finite or countably infinite number of jumps. The corresponding density function then is given as

$$f(x) = \sum_i P_i \delta(x - x_i), \qquad P_i = P\{x = x_i\}.$$

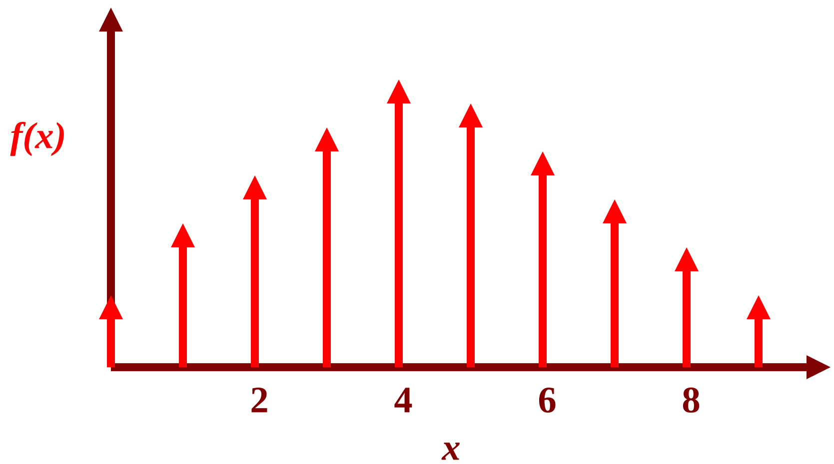


Figure 1. A typical probability density function of a discrete random variable.

Properties of $f_X(x)$

Important properties of the probability density function are:

i) $f(x) \geq 0$.

ii) $\int_{-\infty}^{+\infty} f(x)dx = 1$.

iii) $F(x) = \int_{-\infty}^{x} f(\xi)d\xi$.

iv) $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f(x)dx$

v) For a continuous random variable, $P\{x < X \leq x + \Delta x\} \approx f(x)\Delta x$ and

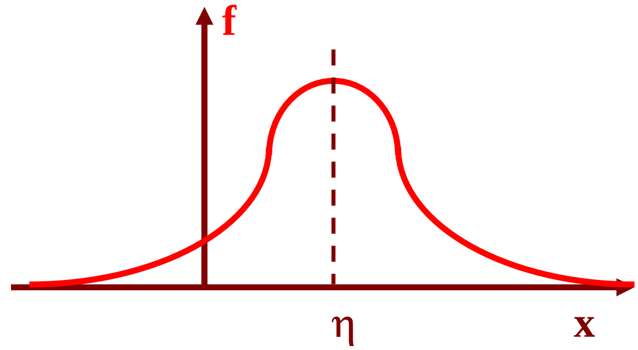
$$f(x) = \lim_{\Delta x \to 0} \frac{P(x < X \leq x + \Delta x)}{\Delta x}$$

Common Probability Density Functions

Normal

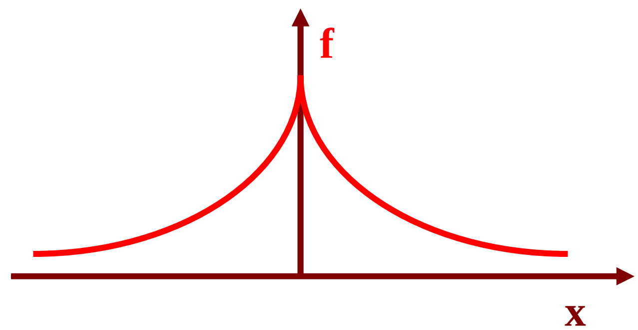
$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\eta)^2}{2\sigma^2}},$$

$$F_x(x) = \frac{1}{2} + erf\,\frac{x-\eta}{\sigma}$$



Laplace

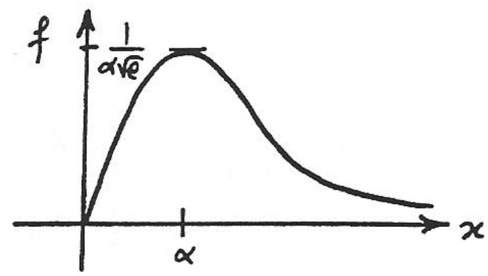
$$f(x) = \frac{\alpha}{2} e^{-\alpha|x|}$$



Rayleigh

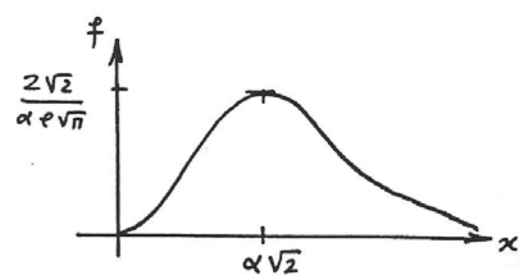
$$f(x) = \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} U(x)$$

$$U(x) = \begin{Bmatrix} 1 & x \geq 0 \\ 0 & x < 0 \end{Bmatrix}$$



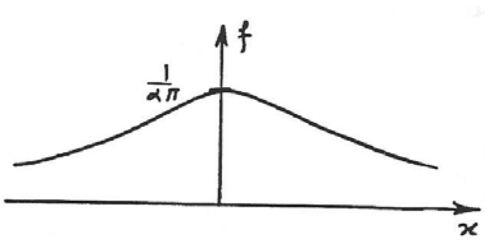
Maxwell

$$f(x) = \frac{\sqrt{2}}{\sqrt{\pi}\alpha^3} x^2 e^{-\frac{x^2}{2\alpha^2}}$$



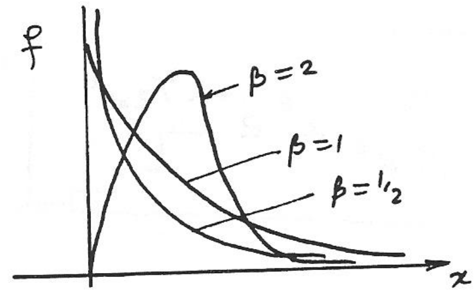
Cauchy

$$f(x) = \frac{\frac{\alpha}{\pi}}{\alpha^2 + x^2}$$



Weibull

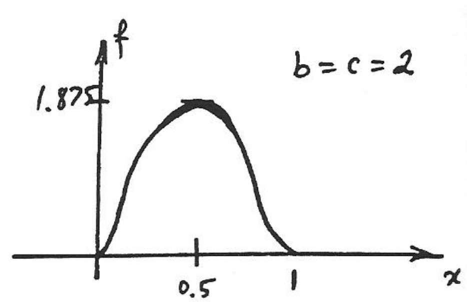
$$f_x(x) = \begin{cases} kx^{\beta-1}e^{-\alpha x^{\beta}} & x > 0 \\ 0 & otherwise \end{cases}$$



Beta

$$f(x) = \begin{cases} Ax^{b}(1-x)^{c} & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

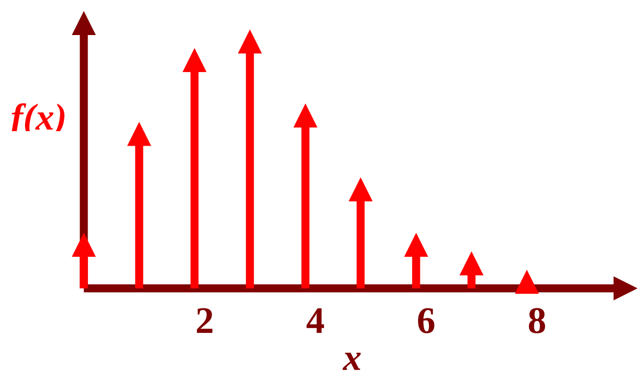
$$A = \frac{\Gamma(b+c+2)}{\Gamma(b+1)\Gamma(c+1)}$$

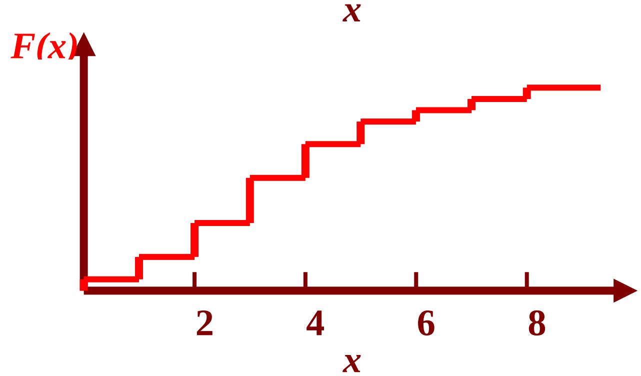


Poisson

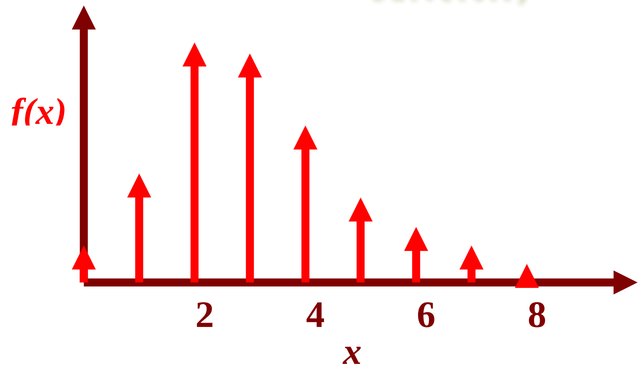
$$P\{x(\xi)=k\} = e^{-a}\frac{a^k}{k!}$$

$$f(x) = e^{-a}\sum_{k=0}^{\infty}\frac{a^k}{k!}\delta(x-k)$$





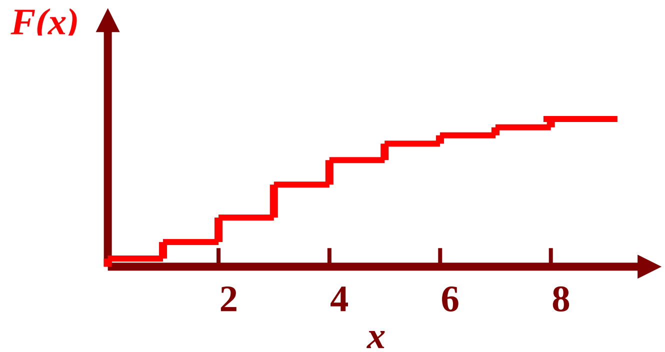
Binomial



$$P\{x=k\}=\binom{n}{k}p^{k}q^{n-k}$$

$$f(x)=\sum_{k=0}^{n}\binom{n}{k}p^{k}q^{n-k}\delta(x-k)$$

$$p+q=1$$



Erlang

$$f(x)=\frac{c^{n}}{(n-1)!}x^{n-1}e^{-cx}U(x)$$

Gamma

$$f(x)=\frac{c^{b+1}}{\Gamma(b+1)}x^{b}e^{-cx}U(x), \qquad \Gamma(b)=\int_{0}^{\infty}y^{b-1}e^{-y}dy$$

Exponential

$$f(x)=ce^{-cx}U(x).$$