ME 529 - Stochastics

Bernoulli Trials

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Bernoulli Trials

Outline

- Repeated Trials
- Most Likely Number of Success
- Asymptotic Theorems
- Gaussian Functions
- Generalized Bernoulli Trials
- Poisson Theorem
- Random Poisson Points

Bernoulli Trials

Series of Independent Experiments

$P(a) = p$ $\quad P(\bar{a}) = q$ $\quad p + q = 1$

Probability that event *a* occurs k times in specific order in n trials:

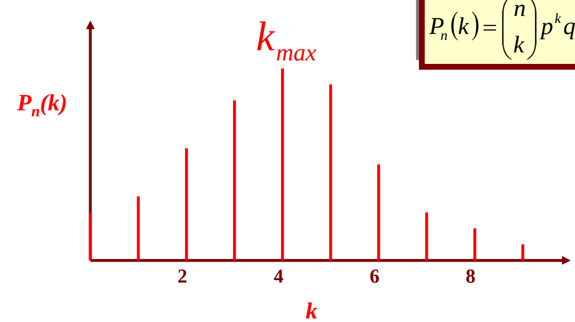
$$P_n(k)\big|_{Specific\,order} = p^k q^{n-k}$$

Probability that event *a* occurs k times in n trials:

$$P_n(k) = \binom{n}{k} p^k q^{n-k}$$

Bernoulli Trials

$$P_n(k) = \binom{n}{k} p^k q^{n-k}$$



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Most Likely Number of Success

$$k_{\max} = \begin{cases} k_1 & k_1 = Greatest\ Integer \le (n+1)P \quad if\ (n+1)P \ne Integer \\ k_1\ and\ k_1 - 1 & k_1 = (n+1)P \quad if\ (n+1)P = Integer \end{cases}$$

$$P(k_1 \le k \le k_2) = \sum_{k=k_1}^{k_2} \binom{n}{k} p^k q^{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n = 1$$

Asymptotic Theorems

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DeMoivre – Laplace Theorem

For n→∞, npq>>1 $k \sim (npq)^{1/2}$ nbh of np

$$P_n(k) = \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} e^{-\frac{(k-np)^2}{2npq}}$$

Stirling Formula

$$n! = n^n e^{-n} \sqrt{2\pi n}$$

$$P_n(k_1 \le k \le k_2) = \sum_{k=k_1}^{k=k_2} \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} \sum_{k=k_1}^{k=k_2} e^{-\frac{(k-np)^2}{2npq}}$$

Asymptotic Theorems

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$$P_n(k_1 \le k \le k_2) \approx \frac{1}{\sqrt{2\pi npq}} \int_{k=k_1}^{k=k_2} e^{-\frac{(x-np)^2}{2npq}} dx$$

Approximate Evaluation

$$P_n(k_1 \le k \le k_2) \approx erf\,\frac{k_2 - np}{\sqrt{npq}} - erf\,\frac{k_1 - np}{\sqrt{npq}}$$

$$erf(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{y^2}{2}} dy \qquad erf(\infty) = \frac{1}{2}$$

Gaussian Functions

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$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad G(x) = \int_{-\infty}^{x} g(y)dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

g, G

$$P_n(k) = \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{npq}} g\left(\frac{k-np}{\sqrt{npq}}\right)$$

$$P_n(k_1 \le k \le k_2) = G\left(\frac{k_2 - np}{\sqrt{npq}}\right) - G\left(\frac{k_1 - np}{\sqrt{npq}}\right)$$

Generalized Bernoulli Trials

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For $a_i \cap a_j = 0$ **With** $P(a_r) = p_r$

$$P_n(k_1, k_2, \ldots, k_r) = \frac{n!}{k_1!\, k_2! \ldots k_r!} p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$$

DeMoivre – Laplace

$$P_n(k_1, k_2, \ldots, k_r) \approx \frac{exp\left\{-\frac{1}{2}\left[\frac{(k_1 - np_1)^2}{np_1} + \ldots + \frac{(k_r - np_r)^2}{np_r}\right]\right\}}{\sqrt{(2\pi n)^{r-1} p_1 \ldots p_r}}$$

Poisson Theorem

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For large n, small p with np being finite

$$P_n(k) = \binom{n}{k} p^k q^{n-k} \approx e^{-nP} \frac{(np)^k}{k!} = e^{-a} \frac{a^k}{k!}$$

$$P_n(k_1 \le k \le k_2) \approx e^{-nP} \sum_{k=k_1}^{k_2} \frac{(np)^k}{k!}$$

Random Poisson Points

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We place at random n points in the interval (0,T). Let $t_2 - t_1 = t_a$. The probability of finding k points in t_a is

$$P(k \text{ Points in } t_a) = \binom{n}{k} p^k q^{n-k} \qquad p = \frac{t_a}{T}$$

Poisson

$$P(k \text{ Points in } t_a) \approx e^{-\lambda t_a} \frac{(\lambda t_a)^k}{k!} \qquad \lambda = \frac{n}{T}$$

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Concluding Remarks

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