Review of Set Theory and Probability Space

i) Set

A set is a collection of objects. These objects are called elements of the set.

ii) Subset

A subset b of a set a is a set whose elements are also elements of a.

iii) Space

"Space" S is the largest set and all other sets under consideration are subsets of S.

iv) Null Set

O is an empty or null set. O contains no elements.

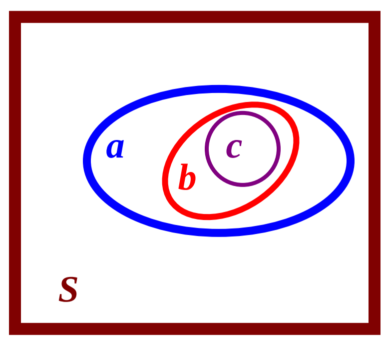
Set Operations

A set b is a subset of a, $b \subset a$, or the set a contains b, $a \supset b$, if all elements of b are also elements of a. That is,

If $b \subset a$, and $c \subset b$, then $c \subset a$.

The following relationship holds:

$$a \subset a, \; 0 \subset a, \; a \subset S$$



Examples of subsets.

i) Equality

$$a = b \text{ iff } a \subset b \text{ and } b \subset a.$$

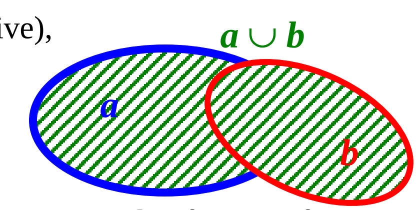
ii)Union (Sum)

The union of two sets a and b is a set consisting of all elements of a or of b or of both. The union operation satisfies the following properties:

$a \cup b = b \cup a$, (Commutative),

$a \cup a = a$,

$a \cup 0 = a$,



An example of union of sets a and b.

$$a \cup S = S,$$

$$(a \cup b) \cup c = a \cup (b \cup c) = a \cup b \cup c, \qquad \text{(Associative).}$$

iii) Intersection (Product)

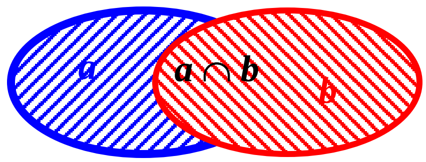
The intersection of two sets a and b is a set consisting of all elements that are common to the sets a and b. The intersection operation satisfies the following properties:

$$a \cap b = b \cap a, \qquad \text{(Commutative),}$$

$$a \cap a = a,$$

$$a \cap 0 = 0,$$

$$a \cap S = a,$$



An example of intersection of sets a and b.

$$(a \cap b) \cap c = a \cap (b \cap c) = a \cap b \cap c, \qquad \text{(Associative),}$$

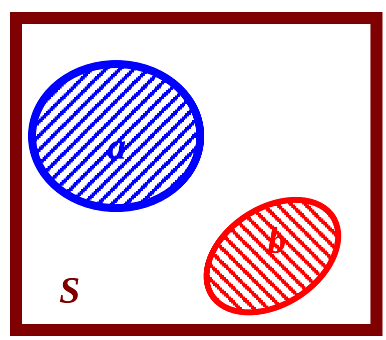
If $b \subset a$, $\quad b \cap a = b$,

Also $\quad a \cap (b \cup c) = (a \cap b) \cup (a \cap c), \qquad$ (Distributive).

Mutually Exclusive Sets

Two sets a and b are called mutually exclusive or disjoint if they have no common elements, i.e.

$$a \cap b = 0.$$



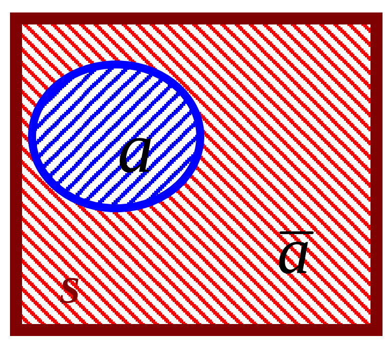
An example of mutually exclusive sets.

The sets a_1, a_2, … are called mutually exclusive if $a_i \cap a_j = 0$ for every $i \neq j$.

Complements

The complement $\bar{a}$ of a set a is defined as a set consisting of all elements of S that are not in a. Complement sets satisfy the following properties:

$$a \cup \bar{a} = S,$$



An example of complemets.

$a \cap \bar{a} = 0$,

$\bar{0} = S$, $\quad \bar{S} = 0$,

If $b \subset a$, $\quad \bar{b} \supset \bar{a}$.

De Morgan Law

$\overline{a \cup b} = \bar{a} \cap \bar{b}$, $\quad \overline{a \cap b} = \bar{a} \cup \bar{b}$.

Difference of Two Sets

The difference set of $a - b$ is a set consisting of elements of a that are not in b. The difference satisfy the following properties:

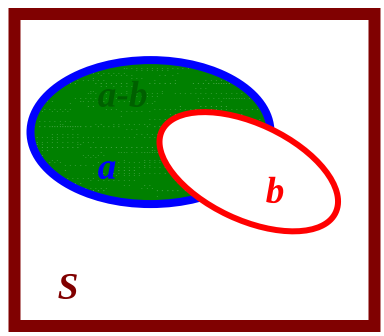
$a - b = a \cap \bar{b} = a - a \cap b$,

$a \cup a - a = 0$,

$(a - a) \cup a = a$,

$\bar{a} = S - a$,

$a = (a - b) \cup (a \cap b)$.



An example of difference of two sets.

Probability space

i) Random Experiment $\Im$

By an experiment $\Im$, we mean a (set) space S of outcomes ξ. Elements of S are *outcomes* or *elementary events*. S is a probability (sample) space. Subsets of S are called *events*. Space S is the *sure (certain) event*. Empty set 0 is the *impossible event*.

ii) Mutually Exclusive Events

Two events a and b are mutually exclusive if $a \cap b = 0$.

iii) Axioms of Probability

To each event a a measure (number) $P(a)$ which is called the *probability of event* a is assigned. $P(a)$ is subjected to the following three axioms:

1. $P(a) \geq 0$,

2. $P(S)=1$,
3. If $a \cap b = 0$, then $P(a \cup b) = P(a) + P(b)$.

Corollaries

$P(0) = 0$,

$P(a) = 1 - P(\bar{a}) \leq 1$.

If $a \cap b \neq 0$, then $P(a \cup b) = P(a) + P(b) - P(a \cap b)$.

If $b \subset a$, $\quad P(a) = P(b) + P(a \cap \bar{b}) \geq P(b)$.

Field

Def: A field F is a nonempty class of sets such that

1. If $a \in F$, then $\bar{a} \in F$;
2. If $a \in F$ and $b \in F$, then $a \cup b \in F$.

Corollaries

If $a \in F$ and $b \in F$, then $a \cap b \in F$ and $a - b \in F$.

Also, $0 \in F$ and $S \in F$.

Borel Field

Def: If a field has the property that if the sets a_1, a_2, … , a_n, … belong to it, then so does the set $a_1 \cup a_2 \cup ... \cup a_n \cup ...$, then the field is called a Borel field. Note that the class of all subsets of S is Borel field.

Probability Experiment $\Im$

A probability experiment is:

1. A set S of outcomes ξ; this set is called space or sure (certain) event.
2. A Borel field F consisting of certain subsets of S called events.
3. A measure (number) $P(a)$ assigned to every event a; This measure is called probability of event a, it satisfies axioms 1-3.

It is common to use the following notation for probability experiments:

$\Im:(S,F,P)$ identifies a probability experiment with space of outcomes *S*, and the associated field *F* with *P(a)* for all outcomes assigned.

Example: Probability experiment of tossing a coin, $\Im:(S,F,P)$. Here the space is

$$S=\{h,t\}.$$

The events are:

$$F:\ 0\,,\ \{h\},\ \{t\},\ \{h,t\},$$

with the probability of the events given as:

$$P(h)=p\,,\ P\{t\}=q\,,\ p+q=1\,.$$