Review of Engineering Mathematics

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- Special Functions
- Differential Equations
- Fourier Series and Transforms
- Probability and Random Processes
- Linear System Analysis

Special Functions

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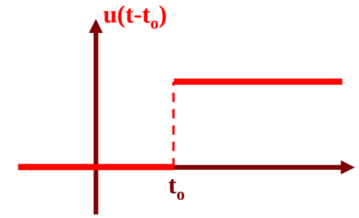
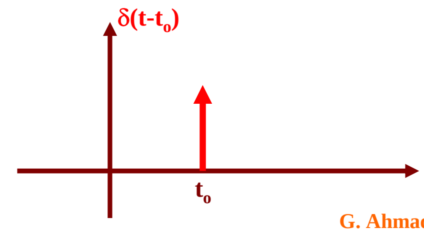
Unit Step Function

$$u(t-t_0)=\begin{cases}1 & t\ge t_0\\ 0 & t<t_0\end{cases}$$



Dirac Delta Function

$$\delta(t-t_0)=\frac{du(t-t_0)}{dt}$$



Properties of Delta Function

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$$\int_{-\infty}^{+\infty}\delta(t-t_0)dt=\int_{t_0-\varepsilon}^{t_0+\varepsilon}\delta(t-t_0)dt=1$$

$$\int_{-\infty}^{+\infty}f(t)\delta(t-t_0)dt=\int_{t_0-\varepsilon}^{t_0+\varepsilon}f(t)\delta(t-t_0)dt=f(t_0)$$

$$\int_{-\infty}^{t}f(t_1)\delta(t_1-t_0)dt_1=f(t_0)u(t-t_0)$$

$$\delta[a(t-t_0)]=\frac{1}{|a|}\delta(t-t_0)$$

Special Functions

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Error Function

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

$$\operatorname{erfc}(x)=1-\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

$$\operatorname{erf}(0)=0=\operatorname{erfc}(\infty)$$

$$\operatorname{erf}(-x)=-\operatorname{erf}(x)$$

Exponential Integrals

$$E_i(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$$

$$E_n(x)=\int_1^\infty\frac{e^{-xt}}{t}dt$$

Fourier Transforms

Fourier Integral Representation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\omega(x-x')} f(x')dx'd\omega$$

Fourier Transform (Exponential)

$$\bar{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x'} f(x')dx'$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega x}\bar{f}(\omega)d\omega$$

Fourier Transform of Derivatives

$$\Im\left\{\frac{df}{dx}\right\} = \int_{-\infty}^{+\infty} e^{-i\omega x}\frac{df(x)}{dx}dx = i\omega\bar{f}(\omega)$$

$$\Im\left\{\frac{d^2f}{dx^2}\right\} = -\omega^2\bar{f}(\omega)$$

$$\Im\left\{\frac{d^nf}{dx^n}\right\} = (i\omega)^n\bar{f}(\omega)$$

Example

$$\frac{d^2f}{dx^2} + a\frac{df}{dx} + bf = \delta(x - x_0)$$

$$-\infty < x < +\infty$$

Taking Fourier Transform

$$-\omega^2\bar{f}(\omega) + ai\omega\bar{f}(\omega) + b\bar{f}(\omega) = e^{-i\omega x_0}$$

$$\bar{f}(\omega) = \frac{e^{-i\omega x_0}}{b - \omega^2 + ia\omega}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\omega(x-x_0)}}{b - \omega^2 + ia\omega}d\omega$$

Table of Fourier Transform

$f(x)$	$\bar{f}(\omega)$
$f_1(x + x_0)$	$e^{i\omega x_0}\bar{f}(\omega)$
$\delta(x - x_0)$	$e^{-i\omega x_0}$
$e^{-\alpha\lvert x\rvert}$	$\frac{2\alpha}{\omega^2 + \alpha^2}$
$f_1(x) * f_2(x) = \int_{-\infty}^{+\infty} f_1(\xi) f_2(x - \xi)d\xi$	$\bar{f}_1(\omega)\bar{f}_2(\omega)$
$\cos\omega_0 x$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$e^{-\alpha\lvert x\rvert}\cos\beta x$	$\frac{2\alpha(\omega^2 + \alpha^2 + \beta^2)}{(\omega^2 - \beta^2 - \alpha^2)^2 + 4\alpha^2\omega^2}$
$e^{-\alpha\lvert x\rvert}\left[\cos\beta x + \frac{\alpha}{\beta}\sin\beta\lvert x\rvert\right]$	$\frac{4\alpha(\alpha^2 + \beta^2)}{(\omega^2 - \beta^2 - \alpha^2)^2 + 4\alpha^2\omega^2}$

Table of Fourier Transform

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$f(x)$	$\bar{f}(\omega)$
$e^{-\alpha^2 x^2}\cos\beta x$	$\frac{\sqrt{\pi}}{2\alpha}\left[\exp\left\{-\frac{(\omega+\beta)^2}{4\alpha^2}\right\}+\exp\left\{-\frac{(\omega-\beta)^2}{4\alpha^2}\right\}\right]$
$e^{-\alpha^2 x^2}$	$\frac{\sqrt{\pi}}{\alpha}\exp\left\{-\frac{\omega^2}{4\alpha^2}\right\}$
$\frac{d^n}{dx^n}\delta(x)$	$(i\omega)^n$
$J_0(x)$	$\begin{cases}\frac{2}{\sqrt{1-\omega^2}} & \lvert\omega\rvert<1 \\ 0 & \text{elsewhere}\end{cases}$

Probability and Random Processes

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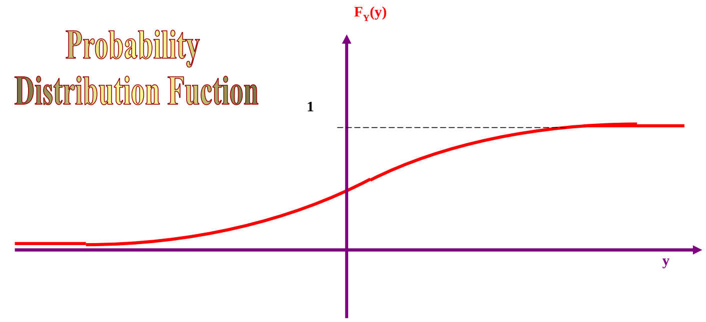
Probability Distribution Fuction



$$F_Y(y) = P\{Y \le y\}$$

$$0 \le F_Y(y) \le 1$$

Probability Density Function

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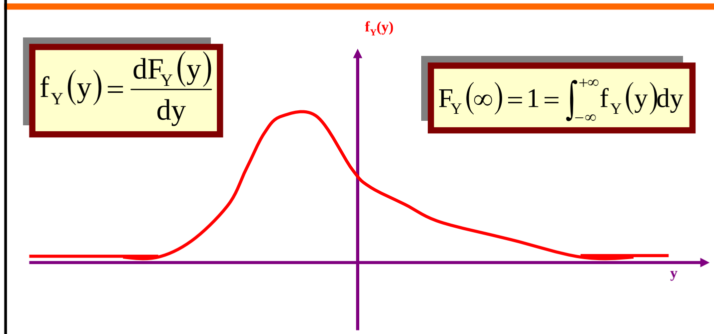
$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$F_Y(\infty) = 1 = \int_{-\infty}^{+\infty} f_Y(y)\,dy$$



$$P\{y_1 < Y \le y_2\} = \int_{y_1}^{y_2} f_Y(y)\,dy = F_Y(y_2) - F_Y(y_1)$$

Probability and Random Processes

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Expected Value

$$E\{Y\} = \bar{Y} = \int_{-\infty}^{+\infty} y f_Y(y)\,dy$$

$$E\{g(Y)\} = \overline{g(Y)} = \int_{-\infty}^{+\infty} g(y) f_Y(y)\,dy$$

Variance

$$\sigma_Y^2 = E\left\{(Y-\bar{Y})^2\right\} = E\{Y^2\} - \bar{Y}^2$$

Stochastic Processes

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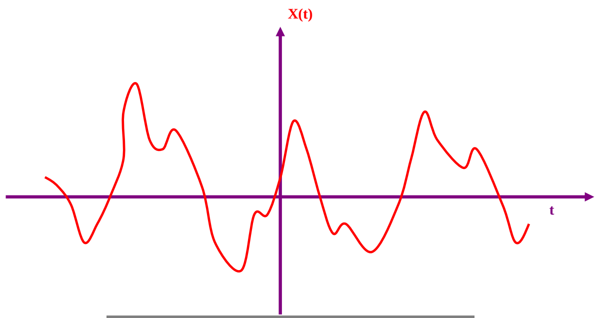

$$E\{X(t)\} = \int_{-\infty}^{+\infty} x f_X(x,t)\,dx$$

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Stochastic Processes

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Time Averaging

$$\overline{X}(t) = \frac{1}{T}\int_0^T X(t)\,dt \approx E\{X(t)\}$$

Autocorrelation

$$R_{xx}(\tau) = E\{X(t+\tau)X(t)\} = \frac{1}{T}\int_0^T X(t+\tau)X(t)\,dt$$

$$R_{xx}(0) = E\{X^2(t)\} = \overline{X^2(t)}$$

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Energy Spectrum

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$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau} R_{xx}(\tau)\,d\tau$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega\tau} S_{xx}(\omega)\,d\omega$$

$$S_{xx}(\omega) = \frac{1}{T}\left|\tilde{X}(\omega)\right|^2$$

$$\tilde{X}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} X(t)\,dt$$

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Linear Systems Analysis

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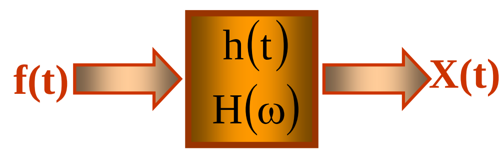
h(t)=Impulse Response

H(ω)=System Function



$$X(t) = \int_0^t h(t-\tau) f(\tau)\,d\tau$$

$$H(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} h(t)\,dt$$

$$X(t) = \int_{-\infty}^{+\infty} h(t-\tau) f(\tau)\,d\tau = h(t) * f(t)$$

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Linear Systems Analysis

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Fourier Transform → $\tilde{x}(\omega) = H(\omega)\tilde{f}(\omega)$

$$S_{xx}(\omega) = \frac{1}{T}\left|\bar{x}(\omega)\right|^2 = \frac{1}{T}\left|H(\omega)\right|^2\left|\bar{f}(\omega)\right|^2 = \left|H(\omega)\right|^2 S_{ff}(\omega)$$

Spectral Relationship

$$S_{xx}(\omega) = \left|H(\omega)\right|^2 S_{ff}(\omega)$$

Linear Systems Analysis - Impulse Response

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$\dot{x} + \alpha x = f(t)$ → $h(t) = e^{-\alpha t}$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = f(t)$$

$$h(t) = \frac{1}{\omega_d} e^{-\zeta\omega_0 t}\sin\omega_d t \qquad \omega_d = \omega_0\sqrt{1-\zeta^2}$$

Numerical Solution of Differential Equations

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$\dot{x} + \alpha x = n(t)$ → Finite difference → $\frac{x_{i+1} - x_i}{\Delta t} + \alpha x_{i+1} = n_i$

$x_i = x(t_i)$

$$x_{i+1} = \frac{1}{1+\alpha\Delta t}x_i + \frac{\Delta t}{1+\alpha\Delta t}n_i$$

$$x_{i+1} = x_i + \Delta t\, n_i \quad \text{for} \quad \alpha\Delta t << 1$$

$$x_{i+1} = \frac{1}{\alpha\Delta t}x_i + \frac{1}{\alpha}n_i \quad \text{for} \quad \alpha\Delta t >> 1$$

White Noise

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