Second Order Modeling of Turbulence

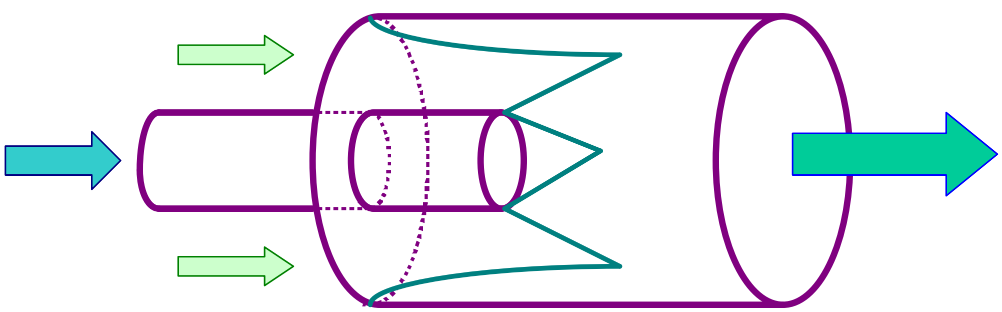
Roughly speaking, if turbulent is characterized by a single length and a single velocity scale, first order modeling (the mixing length and related models) is expected to give reasonable results. The mechanism of transport is superficially like that of turbulence, but the total amount of transport is reasonable estimated. (This is because the constants in the model are calibrated against the data.)

First order modeling breaks down completely in many situations, when there are more than one length or velocity scales. In these situations, the mixing length type models cannot predict the fluxes even approximately. A typical example is the buoyancy driven surface mixing layer where heat flux can occur in the opposite direction of the temperature gradient.

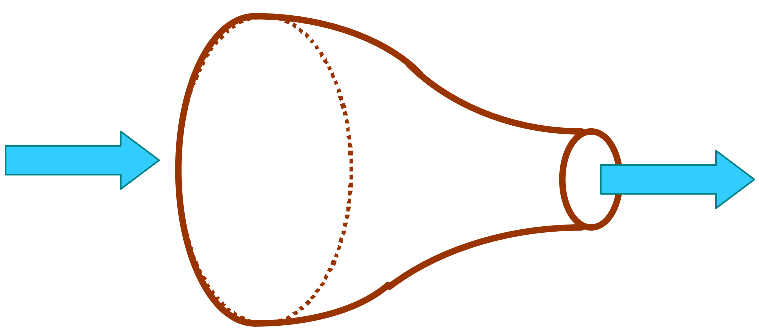
Second order models are expected to work in the situations in which the first order models are not applicable. This expectation is due to the fact that many terms, which are responsible for various mechanisms are carried through. However, past experience shows that when the first order models work, the second order models do not give much better results.

Two-Equation Turbulence Models

Typical examples where the multi-equation turbulence models are needed are shown in the figures.



Multiple scales.



Accelerated flows.

In accordance with the Prandtl-Kolmogorov equation,

$$\nu_T = k^{\frac{1}{2}}\ell \tag{1}$$

where k is the kinetic energy of turbulence and ℓ is the turbulence length scale. While the transport equation for k is well known, a transport equation for ℓ is needed. Usually, a transport equation for a combination of k and ℓ is formulated. Let

$$z = k^m \ell^n. \tag{2}$$

Different authors have used different choices for z that are listed in the table in the past.

Table 1. Commonly used choices for z.

Author	z	Symbol	m	n
Kolmogorov (1942)	$\frac{k^{\frac{1}{2}}}{\ell}$	f or w	$\frac{1}{2}$	-1
Chou (1945), Jones and Launder (1972), Launder and Spalding (1972)	$\frac{k^{\frac{3}{2}}}{\ell}$	ε	$\frac{3}{2}$	-1
Rotta (1951)	ℓ	ℓ	0	1
Rotta (1968), Ng-Spalding (1972)	$k\ell$	$k\ell$	1	1
Spalding (1969)	$\frac{k}{\ell^2}$	W	1	-2

For a thin shear layer, the k-equation is given as

$$\frac{dk}{dt} = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial y}\right) + k\left[\frac{\nu_T}{k}\left(\frac{\partial U}{\partial y}\right)^2 - c_D\frac{k}{\nu_T}\right], \tag{3}$$

with ν_T given is by (1). The general transport equation for z is given as

$$\frac{dz}{dt} = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_z}\frac{\partial z}{\partial y}\right) + z\left[C_1\frac{\nu_T}{k}\left(\frac{\partial U}{\partial y}\right)^2 - C_2\frac{k}{\nu_T}\right] + \frac{s_z}{\rho}, \tag{4}$$

where σ_z, C_1, and C_2 are constants ($\sigma_z \approx 1$).

From the data for decay of turbulence behind a grid, we know that k decays as x^{-1}. Equations (3) and (4) become

$$U_0\frac{dk}{dx} = -C_D\frac{k^2}{\nu_T}, \tag{5}$$

$$U_0\frac{dz}{dx} = -C_2\frac{kz}{\nu_T}. \tag{6}$$

Compatibility of Equations (5) and (6) implies that

$$C_2 = C_D(m - \frac{n}{2}). \tag{7}$$

The constant C_1 may be estimated from matching with limiting flow in the inertial sublayer near a wall. That is, in the inertial layer,

$$\ell = \kappa y, \qquad k = C_D^{-\frac{1}{2}}u^{*2}, \quad \nu_T = C_D^{-\frac{1}{2}}\kappa u^* y, \tag{8}$$

and

$$z = k^m\ell^n = C_D^{-\frac{m}{2}}u^{*2m}(\kappa y)^n = Ay^n, \tag{9}$$

where A is a constant.

Equations (3) and (4) may now be restated as

$$\frac{\nu_T}{k}(\frac{\partial U}{\partial y})^2 - C_D\frac{k}{\nu_T} = 0, \tag{10}$$

$$\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_z}\frac{\partial z}{\partial y}\right) + z\left[C_1\frac{\nu_T}{k}(\frac{\partial U}{\partial y})^2 - C_2\frac{k}{\nu_T}\right] = 0. \tag{11}$$

Eliminating $\frac{\partial U}{\partial y}$, we find

$$\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_z}\frac{\partial z}{\partial y}\right)+\frac{zk}{\nu_T}(C_1C_D - C_2) = 0\,. \qquad (12)$$

Using (8) and (9) in (12) and rearranging, we find

$$C_1 = \frac{C_2}{C_D} - \frac{\kappa^2 n^2}{\sigma_z C_D} \qquad (13)$$

For the ε-equation, ($m = \frac{3}{2}$, $n = -1$) equation (7) and (13) gives

$$C_2 = 2C_D\,, \qquad C_1 = 2 - \frac{\kappa^2}{\sigma_z C_D}\,. \qquad (14)$$

Several z-equations are given in the following section.

Final z-equations, Launder and Spalding (1972)

The following z-equations were suggested by Launder and Sparlding (1972):

$k\ell$ - *Equation*

$$\frac{D}{Dt}(k\ell) = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_{k\ell}}\frac{\partial (k\ell)}{\partial y}\right) + 0.98k^{\frac{1}{2}}\ell^2\left(\frac{\partial U}{\partial y}\right)^2 - 0.059k^{\frac{3}{2}} - \underbrace{\left[702\left(\frac{\ell}{y}\right)^6 k^{\frac{3}{2}}\right]}_{\text{For Near Wall Flows}}, \qquad (15)$$

Here $\sigma_k = 1$, $\sigma_{k\ell} = 1$, and $c_D = 0.09$.

$W = \frac{k}{\ell^2}$ -*Equation*

$$\frac{D}{Dt}W = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_W}\frac{\partial W}{\partial y}\right) + 1.04W^{\frac{1}{2}}\left(\frac{\partial U}{\partial y}\right)^2 - 0.17W^{\frac{3}{2}} + 3.5\nu_T\left(\frac{\partial^2 U}{\partial y^2}\right)^2, \qquad (16)$$

where, $\sigma_k = 0.9$, $\sigma_w = 0.9$, and $c_D = 0.09$.

ε-*Equation*

$$\frac{D}{Dt}\varepsilon = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) + 1.45k\left(\frac{\partial U}{\partial y}\right)^2 - 0.18\frac{\varepsilon^2}{k}, \tag{17}$$

where $\sigma_k = 1$, $\sigma_\varepsilon = 1.3$, and $c_D = 0.09$.

Boundary Conditions

The appropriate boundary conditions are discussed in this section.

At Plane or Axis of Symmetry

$$\frac{\partial k}{\partial y} = 0, \qquad \frac{\partial z}{\partial y} = 0 \tag{18}$$

At Free Surface

The limiting forms of equations (3) and (4) imply that

$$U_0\frac{dk_0}{dx} = c_D\frac{k_0^2}{\nu_{To}}, \qquad \frac{\partial k_0}{\partial y} = 0, \tag{19}$$

$$U_0\frac{dz_0}{dx} = c_z\frac{k_0 z_0}{\nu_{To}}, \qquad \frac{\partial z_0}{\partial y} = 0 \tag{20}$$

Near a Wall

$$U^+ = \frac{1}{\kappa}\ln y^+ + C \tag{21}$$

$$k^+ = C_D^{-\frac{1}{2}} \tag{22}$$

$$z^+ = C_D^{\frac{1}{2}(\frac{n}{2}-m)}(\kappa y^+)^n \tag{23}$$

The $k-\varepsilon$ Model

As noted before, ε is a special form of the z-function and the equation for ε can be obtained accordingly. Nevertheless, it is instructive to provide a direct derivation for the ε-equation three-dimensional flows.

The exact k-equation is given as

$$\frac{dk}{dt}=-\frac{\partial}{\partial x_i}\left[\overline{u_i'\left(\frac{1}{2}u_j'u_j'+\frac{P'}{\rho}\right)}\right]-\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}-\varepsilon+\nu\nabla^2 k\,. \tag{24}$$

The exact equation for the mean-square flow fluctuation vorticity is given as

$$\frac{d}{dt}\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right)=-\frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{u_j'\omega_i'\omega_i'}\right)+\overline{\omega_i'\omega_j'd_{ij}'}-\nu\overline{\frac{\partial\omega_i'}{\partial x_j}\frac{\partial\omega_i'}{\partial x_j}}\,, \tag{25}$$

where terms of the order of $\left(\frac{u^3}{\Lambda\lambda^2}\right)$ and higher are retained and the smaller order are neglected.

We assume that

$$-\overline{u_i'u_j'}=\nu_T\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)-\frac{2}{3}k\delta_{ij}\,, \tag{26}$$

where

$$\nu_T=C_\mu\frac{k^2}{\varepsilon}\,. \tag{27}$$

Furthermore, let

$$-\overline{u_i'\left(\frac{1}{2}u_j'u_j'+\frac{P'}{\rho}\right)}=\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_i} \tag{28}$$

The k-equation becomes

$$\frac{dk}{dt}=\frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_i}\right)+\nu_T\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}-\varepsilon\,, \tag{29}$$

where the viscous diffusion is neglected.

Recalling that

$$\varepsilon = \nu \overline{\omega_i' \omega_i'}\,, \tag{30}$$

equation (25) (when multiplied by 2ν) is an exact transport equation for ε. Introducing the following closure assumptions:

$$-\nu \overline{u_j' \omega_i' \omega_i'} = \frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_j}\,, \text{ (diffusion)}, \tag{31}$$

$$2\nu \overline{\omega_i' \omega_j' d_{ij}'} = c_{\varepsilon 1} \frac{\varepsilon}{k} \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j}\,, \qquad \text{(production)}, \tag{32}$$

$$2\nu^2 \overline{\frac{\partial \omega_i'}{\partial x_j} \frac{\partial \omega_i'}{\partial x_j}} = c_{\varepsilon 2} \frac{\varepsilon^2}{k}\,, \qquad \text{(dissipation)}, \tag{33}$$

the ε-equation becomes

$$\frac{d\varepsilon}{dt} = \frac{\partial}{\partial x_j} \left(\frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_j} \right) + c_{\varepsilon 1} \nu_T \frac{\varepsilon}{k} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} - c_{\varepsilon 2} \frac{\varepsilon^2}{k}\,. \tag{34}$$

The constants are given as

$c_\mu = 0.09$

$c_{\varepsilon 1} = 1.45$

$c_{\varepsilon 2} = 1.9$

$\sigma_k = 1$

$\sigma_\varepsilon = 1.3$

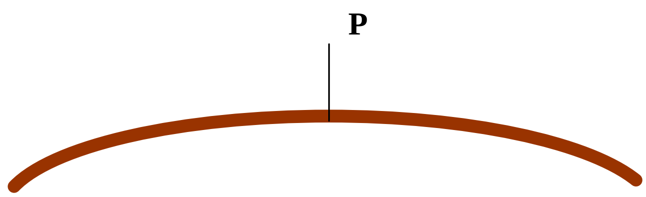
Note that the Reynolds and continuity equation are given as

$$\frac{dU_i}{dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \right] - \frac{2}{3} \frac{\partial k}{\partial x_i}\,, \tag{35}$$

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{36}$$

Equations (29) and (34) - (36) together with ν_T given by (27) form a system of six equations for determining the six unknowns U_i, P, k, and ε.

Boundary Conditions Near a Wall



Schematics of a grid point near a wall.

The velocity boundary condition is given as

$$\frac{U_p}{u^{*2}} C_\mu^{1/4} k_P^{1/2} = \frac{1}{\kappa} \ln\left[E y_p \frac{\left(C_\mu^{1/2} k_p\right)^{1/2}}{\nu} \right], \tag{37}$$

where $E = 9.0$ for a smooth wall. Here k_P is supposed to be known by solving the k-equation. Integrating the k-equation across the grid point, the following assumption is needed:

$$\int_0^{y_P} \varepsilon dy = C_\mu \frac{k_P^{3/2}}{\kappa} \ln\left[\frac{E y_P \left(C_\mu^{1/2} k_P\right)^{1/2}}{\nu} \right] \tag{38}$$

Low Reynolds Number Models

(Jones and Launder (1973), Int. J. Heat Mass Transfer 16, 1119.)

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\nu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_i}\right] + P - \hat{\varepsilon} \tag{39}$$

$$\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_i}\right] + c_{\varepsilon 1}\frac{\hat{\varepsilon}}{k}P + 2\nu\nu_T\left(\frac{\partial^2 U_i}{\partial x_j \partial x_k}\right)^2 - c_{\varepsilon 2}\frac{\hat{\varepsilon}^2}{k} \tag{40}$$

$$\overline{u_i' u_j'} = \frac{2}{3}\delta_{ij}k - \nu_T\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{41}$$

$$\nu_T = c\mu\frac{k^2}{\varepsilon} \tag{42}$$

$$\hat{\varepsilon} = \varepsilon - 2\nu\left(\frac{\partial\sqrt{k}}{\partial x_j}\right)^2 \tag{43}$$

$$c_\mu = 0.09\exp\left\{-\frac{2.51}{(1 + R_T/50)}\right\} \tag{44}$$

$$c_{\varepsilon 2} = 1.9\left[1 - 0.3\exp\left(-R_T^2\right)\right] \tag{45}$$

$$R_T = \frac{k^2}{\nu\varepsilon} \tag{46}$$

$$P = -\overline{u_i' u_j'}\frac{\partial U_i}{\partial x_j} \quad \text{is the production} \tag{47}$$

Modified $k - \varepsilon$ Model for Low Reynolds Number Flows

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

$$U\frac{\partial U}{\partial x}+v\frac{\partial U}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\partial}{\partial y}\left[(\nu+\nu_T)\frac{\partial U}{\partial y}\right]$$

$$\nu_T = c_\mu \frac{k^2}{\varepsilon}$$

$$U\frac{\partial k}{\partial x}+v\frac{\partial k}{\partial y}=\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_T}{\sigma_k}\right)\frac{\partial k}{\partial y}\right]+\nu_T\left(\frac{\partial U}{\partial y}\right)^2-\varepsilon-2\nu\left(\frac{\partial k^{\frac{1}{2}}}{\partial y}\right)^2$$

$$U\frac{\partial \varepsilon}{\partial x}+v\frac{\partial \varepsilon}{\partial y}=\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right]+c_1\frac{\varepsilon}{k}\nu_T\left(\frac{\partial U}{\partial y}\right)^2-c_2\frac{\varepsilon^2}{k}+c_3\nu\nu_T\left(\frac{\partial^2 U}{\partial y^2}\right)^2$$

Here, $c_1 = 1.44$, $c_3 = 2$, $\sigma_k = 1$, $\sigma_\varepsilon = 1.3$, $c_2 = 1.92\left(1-0.3e^{-R_T^2}\right)$,

$C_\mu = 0.09e^{-3.4/\left(1+\frac{R_T}{50}\right)^2}$, $R_T = \frac{k^2}{\nu\varepsilon}$.

Kolmogorov Model

$$\frac{DU_i}{Dt}=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}-\frac{\partial}{\partial x_j}\overline{u_i' u_j'}$$

$$\frac{\partial U_i}{\partial x_i}=0$$

Eddy Diffusivity

$$-\overline{u_i' u_j'}=\nu_T\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)-\frac{2}{3}k\delta_{ij}$$

$$\nu_T=\frac{Ak}{W}$$

$$\frac{Dk}{Dt}=2\nu_T\delta_{ij}^2-\frac{1}{2}k^2W+A''\frac{\partial}{\partial x_j}\left(\frac{k}{W}\frac{\partial k}{\partial x_j}\right),\quad \delta_{ij}=\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)$$

$$\frac{D}{Dt}W=-\frac{7}{10}W^2+2A'\frac{\partial}{\partial x_j}\left(\frac{k}{W}\frac{\partial k}{\partial x_j}\right),$$

where A, A', and A'' are constants and W is the characteristic vorticity.

Saffman Model

$$\nu_T = \frac{Ak}{W}$$

$$\frac{D}{Dt}k = \alpha'' k\left(2S_{ij}^2\right)^{\frac{1}{2}} - kW + A''\frac{\partial}{\partial x_j}\left(\frac{k}{W}\frac{\partial k}{\partial x_j}\right)$$

$$\frac{D}{Dt}W^2 = \alpha' W^2\left[\eta U_{i,j}^2 + 2(1-\eta)S_{ij}^2\right]^{\frac{1}{2}} - \beta' W^3 + A'\frac{\partial}{\partial x_j}\left(\frac{k}{W}\frac{\partial W^2}{\partial x_j}\right),$$

where α', β', α'', A, A', A'', and η are constants.

$\beta' = \frac{5}{3}$, $A = \Gamma^2$, $\alpha'' = \Gamma$, $A' = A'' = \frac{1}{2}A$, $\alpha' = \frac{\beta'\alpha'' - 4A'\kappa^2}{A}$, $\eta = 1$, $\kappa \approx 0.4$ is the Karman constant, $\Gamma = \frac{u^{*2}}{k} \approx 0.3$, and W is the pseudo-vorticity.

Boundary Conditions near a Solid Wall

$$U \sim u^*\left(\frac{1}{\kappa}\ln\frac{yu^*}{\nu} + B\right)$$

$$k \sim \frac{\alpha''}{A}u^{*2}$$

$$W \sim \frac{\alpha'' u^*}{\kappa y}$$

Stress Transport Model for a Two-Dimensional Boundary Layer Flow

The exact equation for $\overline{u'v'}$ in a boundary layer flow is given as

$$\frac{D}{Dt}\overline{u'v'} = -\overline{v'^2}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\left(\overline{u'v'^2} - \frac{\overline{P'u'}}{\rho}\right) + \overline{\frac{P'}{\rho}\left(\frac{\partial u'}{\partial y} + \frac{\partial u'}{\partial x}\right)} - 2\nu\overline{\frac{\partial u'}{\partial x_k}\frac{\partial v'}{\partial x_k}},$$

where $\overline{v'^2}\dfrac{\partial U}{\partial y}$ is the production, $\dfrac{\partial}{\partial y}\left(\overline{u'v'^2}-\dfrac{\overline{P'u'}}{\rho}\right)$ is the diffusion, $\overline{\dfrac{P'}{\rho}\left(\dfrac{\partial u'}{\partial y}+\dfrac{\partial u'}{\partial x}\right)}$ is the pressure-strain, and $2\nu\overline{\dfrac{\partial u'}{\partial x_k}\dfrac{\partial v'}{\partial x_k}}$ is the dissipation.

Modeling (Hanjalic 1970)

Production is approximately equal to $k\dfrac{\partial U}{\partial y}$.

Diffusion is approximately equal to $\dfrac{\partial}{\partial y}\left[\dfrac{\nu_T}{\sigma_T}\dfrac{\partial}{\partial y}\overline{u'v'}\right]$.

Dissipation is approximately equal to 0.

Pressure-strain is approximately equal to $\dfrac{k^{\frac{1}{2}}}{\ell}\overline{u'v'}$.

The Closed transport equation becomes

$$\frac{D}{Dt}\overline{u'v'}=\frac{\partial}{\partial y}\left[\frac{\nu_T}{\sigma_z}\frac{\partial}{\partial y}\overline{u'v'}\right]-c_\tau\left(k\frac{\partial U}{\partial y}+\frac{k^{\frac{1}{2}}}{\ell}\overline{u'v'}\right),$$

where $\sigma_\tau = 0.9$, $c_\tau = 2.8$, and k and ℓ ($\varepsilon = C_D\dfrac{k^{\frac{3}{2}}}{\ell}$) are found from their transport equations.

Harlow and Daly [(1970) Phys. Fluids 13, 2634] obtained five equations for $\overline{u'v'}$, $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$, and ε.

Low-Reynolds-Number

$$U\frac{\partial k}{\partial x}+V\frac{\partial k}{\partial y}=\nu_T\left(\frac{\partial U}{\partial y}\right)^2-\varepsilon+\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_T}{\sigma_k}\right)\frac{\partial k}{\partial y}\right]$$

$$U\frac{\partial \tilde{\varepsilon}}{\partial x}+V\frac{\partial \tilde{\varepsilon}}{\partial y}=C_{\varepsilon 1}f_1\frac{\bar{\varepsilon}}{k}\nu_T\left(\frac{\partial U}{\partial y}\right)^2-C_{\varepsilon 2}f_2\frac{\tilde{\varepsilon}^2}{k}+E+\frac{\partial}{\partial y}\left[\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \tilde{\varepsilon}}{\partial y}\right]$$

$$\varepsilon=\varepsilon_0+\bar{\varepsilon}, \qquad \nu_T=\frac{C_\mu f_\mu k^2}{\tilde{\varepsilon}}$$

$$Re_T=\frac{k^2}{\tilde{\varepsilon}\nu}, \qquad R_y=\frac{k^{\frac{1}{2}}y}{\nu}, \qquad y^+=\frac{u_\tau y}{\nu}$$

$$k \sim y^2 \text{ and } \frac{\varepsilon}{k}\rightarrow\frac{2\nu}{y^2} \text{ as } y\rightarrow 0$$

$$\tau_{xy} \sim y^3$$

Chien Model

$$f_\mu=1-e^{-0.0115y^+}$$

$$f_1=1$$

$$f_2=1-0.22e^{-\left(\frac{Re_T}{6}\right)^2}$$

$$\varepsilon_0=2\nu\frac{k}{y^2}$$

$$E=-2\nu\frac{\tilde{\varepsilon}}{y^2}e^{-\frac{y^+}{2}}$$

$$C_{\varepsilon 1}=1.35, \quad C_{\varepsilon 2}=1.80, \quad C_\mu=0.09, \quad \sigma_k=1.0, \quad \sigma_\varepsilon=1.3$$

$$k=\tilde{\varepsilon}=0 \text{ at } y=0 \qquad \text{(boundary conditions)}$$

Jones-Launder Model

$$f_\mu = e^{-\frac{2.5}{\left(1+\frac{Re_T}{50}\right)}}$$

$$f_1 = 1$$

$$f_2 = 1 - 0.3e^{-Re_T^2}$$

$$\varepsilon_0 = 2v\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2$$

$$E = 2vv_T\left(\frac{\partial^2 U}{\partial y^2}\right)^2$$

$C_{\varepsilon 1} = 1.45$, $C_{\varepsilon 2} = 0.09$, $C_\mu = 0.09$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$

Launder-Sharma Model

$$f_\mu = e^{-\frac{3.4}{\left(1+\frac{Re_T}{50}\right)^2}}$$

$$f_1 = 1$$

$$f_2 = 1 - 0.3e^{-Re_T^2}$$

$$\varepsilon_0 = 2v\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2$$

$$E = 2vv_T\left(\frac{\partial^2 U}{\partial y^2}\right)^2$$

$C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$, $C_\mu = 0.09$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$

Lam-Bremhorst Model

$$f_\mu = \left(1 - e^{-0.0165}R_y\right)^2 \left(1 + \frac{20.5}{Re_T}\right)$$

$$f_1 = 1 + \left(\frac{0.05}{f_\mu}\right)^3$$

$$f_2 = 1 - e^{-e\,Re_T^2}$$

$$\varepsilon_0 = 0$$

$$E = 0$$

$$C_{\varepsilon 1} = 1.44\,, \quad C_{\varepsilon 2} = 1.92\,, \quad C_\mu = 0.09\,, \quad \sigma_k = 1.0\,, \quad \sigma_\varepsilon = 1.3$$

$$\tau_{xy} \sim y^4$$

$$\frac{\partial \varepsilon}{\partial y} = 0 \text{ or } \varepsilon = \nu \frac{\partial^2 k}{\partial y^2} \text{ at } y = 0$$

$k - \omega$ Equation

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left[(\nu + \nu_T)\frac{\partial U}{\partial y}\right]$$

$$U\frac{\partial k}{\partial x} + V\frac{\partial k}{\partial y} = \nu_T\left(\frac{\partial U}{\partial y}\right)^2 - \beta^*\omega k + \frac{\partial}{\partial y}\left[(\nu + \sigma^*\nu_T)\frac{\partial k}{\partial y}\right]$$

$$U\frac{\partial \omega}{\partial x} + V\frac{\partial \omega}{\partial y} = \alpha\frac{\omega}{k}\nu_T\left(\frac{\partial U}{\partial y}\right)^2 - \beta\omega^2 + \frac{\partial}{\partial y}\left[(\nu + \sigma\nu_T)\frac{\partial \omega}{\partial y}\right]$$

$$v_T = \frac{\alpha^* k}{\omega}$$

Or

$$U\frac{\partial k}{\partial x} + V\frac{\partial k}{\partial y} = P_k\beta^*\omega k + \frac{\partial}{\partial y}\left[(\nu + \sigma^*\nu_T)\frac{\partial k}{\partial y}\right]$$

$$U\frac{\partial \omega}{\partial x} + V\frac{\partial \omega}{\partial y} = P_\omega \beta \omega^2 + \frac{\partial}{\partial y}\left[(\nu + \sigma \mu_T)\frac{\partial \omega}{\partial y}\right]$$

$$P_k = \frac{\alpha^*}{\beta^*}\left(\frac{\frac{\partial U}{\partial y}}{\omega}\right)^2 - 1$$

$$P_k = \frac{\alpha \alpha^*}{\beta}\left(\frac{\frac{\partial U}{\partial y}}{\omega}\right)^2 - 1$$

Algebraic Stress Transport Model (Rodi, ZAMM 56 (1976))

A simplified stress transport model is given as

$$\frac{d}{dt}\overline{u_i'u_j'} = c_s \frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_\ell'}\frac{\partial}{\partial x_\ell}\overline{u_i'u_j'}\right) - \overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k} - c_1\frac{\varepsilon}{k}\left(\overline{u_i'u_j'} - \delta_{ij}\frac{2}{3}k\right), \qquad (1)$$
$$-\gamma\left(P_{ij} - \delta_{ij}\frac{2}{3}P\right) - \frac{2}{3}\delta_{ij}\varepsilon$$

where $D_{ij} = \frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_\ell'}\frac{\partial}{\partial x_\ell}\overline{u_i'u_j'}\right)$ is the diffusion, $P_{ij} = -\overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k}$

is the production, $-c_1\frac{\varepsilon}{k}\left(\overline{u_i'u_j'} - \delta_{ij}\frac{2}{3}k\right) - \gamma\left(P_{ij} - \delta_{ij}\frac{2}{3}P\right)$ is the pressure-strain,

and $\frac{2}{3}\delta_{ij}\varepsilon$ is the dissipation.

Here, $P = \frac{1}{2}P_{ii}$ is the production rate of turbulent kinetic energy. Contracting equation (1), we find the transport equation for k:

$$\frac{dk}{dt} = c_s\frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_\ell'}\frac{\partial k}{\partial x_\ell}\right) - \overline{u_k'u_\ell'}\frac{\partial U_k}{\partial x_\ell} - \varepsilon, \qquad (2)$$

where $D = \frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_\ell'}\frac{\partial k}{\partial x_\ell}\right)$ is the diffusion, and $P = \overline{u_k'u_\ell'}\frac{\partial U_k}{\partial x_\ell}$ is the

production.

Radi (1976) assumed that

$$\frac{d}{dt}\overline{u_i'u_j'} - D_{ij} = \frac{\overline{u_i'u_j'}}{k}\left(\frac{dk}{dt} - D\right) = \frac{\overline{u_i'u_j'}}{k}(P - \varepsilon). \qquad (3)$$

Using (3) in (1) and rearranging, the result is

$$\overline{u_i'u_j'} = k\left[\frac{2}{3}\delta_{ij} + \frac{1-\gamma}{c_1}\frac{\frac{P_{ij}}{\varepsilon} - \frac{2}{3}\delta_{ij}\frac{P}{\varepsilon}}{1+\frac{1}{c_1}\left(\frac{P}{\varepsilon}-1\right)}\right]. \tag{4}$$

This is an algebraic expression for $\overline{u_i'u_j'}$.

For simple shear flows, it may be shown that equation (4) reduces to the Kolmogorov-Prandtl hypothesis with

$$\nu_T = c_\mu \frac{k^2}{\varepsilon} \tag{5}$$

and

$$c_\mu = \frac{2}{3}\frac{(1-\gamma)}{c_1}\frac{\left[1-\frac{1}{c_1}\left(1-\gamma\frac{P}{\varepsilon}\right)\right]}{\left[1+\frac{1}{c_1}\left(\frac{P}{\varepsilon}-1\right)\right]^2}. \tag{6}$$