

Correlation, Spectrum, and Scales

Definition: Correlation Tensor (Two-Point)

Consider a turbulent flow field as shown in Figure 1. Here u_1 and u_2 are the components of the velocity vectors and $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ is the distance between the two points. The two-point correlation tensor is defined as

$$Q_{ij}(\mathbf{x}, \mathbf{x}_1) = \overline{u'_i(\mathbf{x})u'_j(\mathbf{x}_1)}$$

In a homogeneous turbulent flow, the correlations (and all the statistics) are independent of the shift of space origin. That is,

$$Q_{ij}(\mathbf{x}, \mathbf{x}_1) = Q_{ij}(\mathbf{r})$$

Definition: Longitudinal Correlation Coefficient

The longitudinal correlation coefficient is defined as

$$f(r) = \frac{Q_{11}}{u_1^2}$$

where

$$Q_{11} = \overline{u'_1(\mathbf{x})u'_1(\mathbf{x}_1)}, \quad u_1^2 = \overline{u_1'^2(\mathbf{x})} = \overline{u_1'^2(\mathbf{x}_1)}$$

Note that $f(r)$ is an even function. That is

$$f(r) = f(-r)$$

A typical longitudinal correlation coefficient is shown in Figure 2.

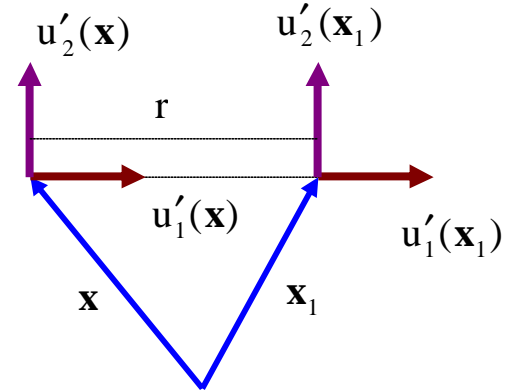


Figure 1. Geometric features of two-point correlations in a turbulent flow field.

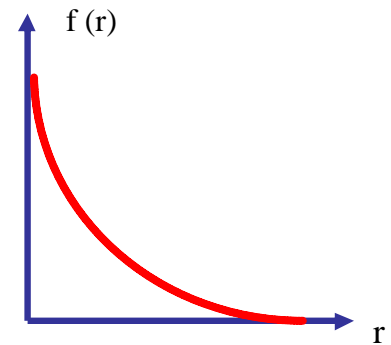


Figure 2. Schematics of longitudinal correlation coefficient.

Definition: Lateral Correlation Coefficient

The lateral correlation coefficient is defines as

$$g(r) = \frac{Q_{22}}{u_2^2}, \quad Q_{22} = \overline{u'_2(\mathbf{x})u'_2(\mathbf{x}_1)}$$

The lateral correlation coefficient is also a symmetric Function. That is

$$g(r) = g(-r)$$

A typical lateral correlation coefficient is shown in Figure 3.

Definition: Taylor's Microscales

The Taylor microscales are defines as

$$\lambda_f^2 = -\frac{2}{f''(0)}, \quad \lambda_g^2 = -\frac{2}{g''(0)},$$

where λ_f and λ_g are, respectively, the Taylor longitudinal and lateral microscales. The microscales may be defined by fitting a parabola to the correlation coefficient curves at $r = 0$. That is,

$$g(r) = 1 + \frac{1}{2!}r^2g''(0) + \dots \approx 1 - \frac{r^2}{\lambda_g^2}$$

Definition: Integral Scales, Macroscales

The macroscales of turbulence are defined as

$$\Lambda_f = \int_0^{\infty} f(r)dr = \text{Longitudinal Macroscale}$$

$$\Lambda_g = \int_0^{\infty} g(r)dr = \text{Lateral Macroscale}$$

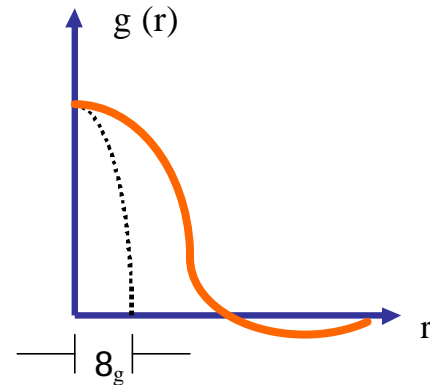


Figure 3. Schematics of lateral correlation function and the corresponding Taylor Scale.

Definition: Eulerian Time Correlation (stationary flows)

The Eulerian time correlation is defined as

$$R_E(\tau) = \frac{\overline{u'_1(\mathbf{x}, t)u'_1(\mathbf{x}, t + \tau)}}{u_1'^2}$$

The Eulerian time microscale τ_E then is given by

$$\tau_E^2 = -\frac{2}{R_E''(0)}$$

The Eulerian time macroscale (integral scale) T_E is defined as

$$T_E = \int_0^{\infty} R_E(\tau) d\tau$$

Using the uniform flow and frozen field approximations, the scales may be related. That is

$$\Lambda_f \approx UT_E, \quad \lambda_f \approx U\tau_E, \quad f(U\tau) \approx R_E(\tau)$$

$$\frac{\partial}{\partial t} = -U \frac{\partial}{\partial x}$$

Definition: Lagrangian Time Correlation

The Lagrangian velocity correlation is defined as

$$R_L(\tau) = \frac{\overline{v'_L(t)v'_L(t + \tau)}}{v_L'^2}$$

where v'_L is the Lagrangian fluctuation velocity. The corresponding Lagrangian time microscale τ_L and the macroscale T_L are given as

$$\tau_L^2 = -\frac{2}{R_L''(0)}$$

and

$$T_L = \int_0^{\infty} R_L(\tau) d\tau$$

Definition: Energy Spectrum Tensor

Energy spectrum of is defined as the Fourier Transform of the correlation tensor. That is

$$E_{ij}(\mathbf{k}) = \frac{1}{8\pi^3} \int_{-\infty-\infty-\infty}^{+\infty+\infty+\infty} Q_{ij}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}$$

$$Q_{ij}(\mathbf{x}) = \int_{-\infty-\infty-\infty}^{+\infty+\infty+\infty} E_{ij}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}$$

Definition: One Dimensional Energy Spectrum

The one dimensional energy spectrum is defined as

$$E_1(k_1) = \frac{u_1^2}{\pi} \int_{-\infty}^{+\infty} f(x_1) e^{-ik_1x_1} dx_1$$

$$u_1^2 f(x_1) = \frac{1}{2} \int_{-\infty}^{+\infty} E_1(k_1) e^{ik_1x_1} dk_1$$

Symmetry of $f(x_1)$ implies that

$$E_1(k_1) = \frac{2u_1^2}{\pi} \int_0^{\infty} f(x_1) \cos k_1x_1 dx_1$$

$$u_1^2 f(x_1) = \int_0^{\infty} E_1(k_1) \cos k_1x_1 dk_1$$

A typical one dimensional energy spectrum is shown in Figure 4.

Setting x_1 equal to zero, we find

$$u_1^2 = \int_0^{\infty} E_1(k_1) dk_1 .$$

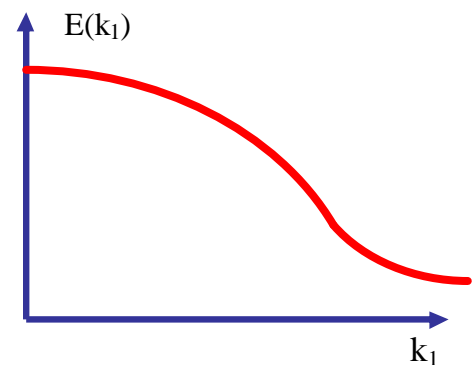


Figure 4. Schematics of one-dimensional energy spectrum.

Also

$$\frac{1}{\lambda_f^2} = -\frac{1}{2} \frac{\partial^2 f}{\partial x_1^2} \Big|_{x_1=0} = \frac{1}{2u_1^2} \int_0^{\infty} k_1^2 E(k_1) dk_1$$

Example: The longitudinal correlation may be approximated as

$$f(r) = e^{-\frac{r}{\Lambda_f}}$$

The corresponding one dimensional spectrum is given as

$$E_1(k_1) = \frac{u_1^2}{\pi} \frac{2\Lambda_f}{1 + \Lambda_f^2 k_1^2}$$

Estimates for Taylor Microscales

The energy dissipation is given as

$$\varepsilon = \nu \overline{\frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j}} \quad (1)$$

For isotropic turbulence, it can be shown that

$$\varepsilon = 15\nu \overline{\left(\frac{\partial u'_1}{\partial x_1}\right)^2} = -15\nu \overline{u_1'^2 f''(0)} \quad (2)$$

It then follows that

$$\varepsilon = 30\nu \frac{\overline{u_1'^2}}{\lambda_f^2} = 30\nu \frac{u^2}{\lambda_f^2} = 15\nu \frac{u^2}{\lambda_g^2} \quad (3)$$

Since

$$\lambda_f^2 = \lambda_g^2 \sqrt{2} \quad (4)$$

Using the macroscopic estimate for the dissipation, we find

$$\varepsilon = A \frac{u^3}{\Lambda} = 30\nu \frac{u^2}{\lambda_f^2} \quad (5)$$

Therefore

$$\frac{\lambda_f}{\Lambda} = \sqrt{\frac{30}{A}} R_\Lambda^{-1/2} \quad (6)$$

Therefore,

$$\frac{\lambda_f}{\Lambda} \ll 1 \quad \text{since} \quad R_\Lambda = \frac{u\Lambda}{\nu} \gg 1 \quad (7)$$

Similarly

$$\frac{\lambda_g}{\Lambda} = \sqrt{\frac{15}{A}} R_\Lambda^{-1/2} \quad (8)$$

It may also be shown that

$$\frac{\lambda_g}{\Lambda} = \frac{15}{A} R_\lambda^{-1}, \quad R_\lambda = \frac{u\lambda}{\nu} \quad (9)$$

and

$$\frac{\lambda_g}{\eta} = \left(\frac{225}{A}\right)^{1/4} R_\Lambda^{-1/4} = 15^{1/4} R_\lambda^{-1/2} \quad (10)$$

From Equation (3) it follows that

$$\frac{u}{\lambda_g} = 0.26 \sqrt{\frac{\varepsilon}{\nu}} = \frac{0.26}{\tau} \quad (11)$$

where the Kolmogorov time scale is given by

$$\tau = \frac{\eta}{\nu} = \sqrt{\frac{\nu}{\varepsilon}} \quad (12)$$

This means that the Taylor microscale is not a characteristic length of the dissipation eddies. It, however, provides a useful artificial length scale for estimating the velocity gradients of the small eddies when macroscopic velocity scale is used for the velocity of the eddies. That is, Equations (1) and (11) imply that

$$\frac{\overline{\partial u'_i}}{\partial x_j} \frac{\overline{\partial u'_i}}{\partial x_j} = \frac{\varepsilon}{\nu} \sim \left(\frac{u}{\lambda_g}\right)^2 \quad (13)$$

Other useful estimates are

$$\overline{d'_{ij} d'_{ij}} \sim \frac{\overline{\partial u'_i}}{\partial x_j} \frac{\overline{\partial u'_i}}{\partial x_j} \sim \left(\frac{u}{\lambda}\right)^2, \quad \frac{u^3}{\Lambda} \sim \nu \frac{u^2}{\lambda^2}, \quad \nu \sim \frac{u\lambda^2}{\Lambda} \quad (14)$$

Also

$$\frac{\lambda}{\Lambda} \sim R_\Lambda^{-1/2} \sim R_\lambda^{-1}, \quad \frac{\eta}{\Lambda} \sim R_\Lambda^{-3/4} \sim R_\lambda^{-3/2}, \quad \frac{\eta}{\lambda} \sim R_\Lambda^{-1/4} \sim R_\lambda^{-1/2} \quad (15)$$

and

$$\eta^2 \Lambda = \lambda^3 \quad (16)$$