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# Boundary Layer-Momentum Integral

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## Boundary Layer

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**Outline**

- **Momentum Integral Method**
- **Momentum Integral for Flat Plate**
- **Momentum Integral with Pressure Gradient**
- **Walz Approximation**



## Boundary Layer over a Flat Plate

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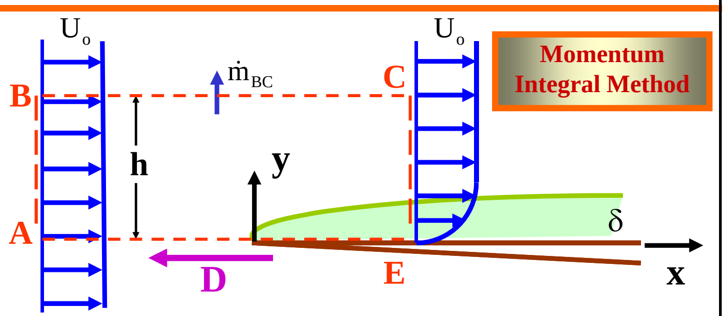


**Momentum Integral Method**

**Conservation of Mass**

$$\rho\int_0^h u\,dy + \dot{m}_{BC} - \rho\int_0^h U_o\,dy = 0$$

$$\dot{m}_{BC} = \rho\int_0^h (U_o - u)\,dy = \rho U_o \delta^*$$



## Integral Method

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**Momentum Equation**

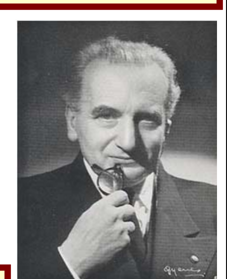
$$\sum_{\text{Outlets}} \rho_o V_o A_o V_{ox} - \sum_{\text{Inlets}} \rho_i V_i A_i V_{ix} = \sum F_x$$

$$\rho\int_0^h u^2\,dy + \dot{m}_{BC} U_o - \rho\int_0^h U_o^2\,dy = -D$$

$$D = \rho\int_0^h u(U_o - u)\,dy = \rho U_o^2 \theta$$

**von Karman Momentum Integral**

$$\tau_w = \rho U_o^2 \frac{d\theta}{dx}$$



von Karman

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## Procedure

- Assume a velocity profile that satisfies the boundary conditions.
- Evaluate wall shear stress and θ.
- Use Momentum Integral and find δ
- Evaluate Boundary Layer parameters θ, δ, $C_F$, $C_D$.

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**With Pressure Gradient**

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad v=-\int_0^y\frac{\partial u}{\partial x}dy$$

$$\int_0^h\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-U\frac{dU}{dx}\right)dy=\nu\int_0^h\frac{\partial^2 u}{\partial y^2}dy=\nu\left.\frac{\partial u}{\partial y}\right|_0^h=-\frac{\tau_w}{\rho}$$

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$$\int_0^h\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-U\frac{dU}{dx}\right)dy=\nu\int_0^h\frac{\partial^2 u}{\partial y^2}dy=\nu\left.\frac{\partial u}{\partial y}\right|_0^h=-\frac{\tau_w}{\rho}$$

The middle term can be restate as

$$\int_0^h v\frac{\partial u}{\partial y}dy=\int_0^h\frac{\partial(uv)}{\partial y}dy-\int_0^h u\frac{\partial v}{\partial y}dy=Uv\,|_{y=h}+\int_0^h u\frac{\partial u}{\partial x}dy$$

$$=-U\int_0^h\frac{\partial u}{\partial x}dy+\int_0^h u\frac{\partial u}{\partial x}dy=-\int_0^h(U-u)\frac{\partial u}{\partial x}dy$$

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$$\int_0^h\left(u\frac{\partial u}{\partial x}-(U-u)\frac{\partial u}{\partial x}-U\frac{dU}{dx}\right)dy=-\frac{\tau_w}{\rho}$$

$$\frac{\tau_w}{\rho}=\int_0^h\left(-u\frac{\partial u}{\partial x}+(U-u)\frac{\partial u}{\partial x}+U\frac{dU}{dx}\right)dy$$

**Rearranging**

$$\frac{\tau_w}{\rho}=\int_0^h\left\{\frac{\partial}{\partial x}[u(U-u)]-u\frac{dU}{dx}+U\frac{dU}{dx}\right\}dy$$

$$\frac{\tau_w}{\rho}=\int_0^h\left\{\frac{\partial}{\partial x}[u(U-u)]+(U-u)\frac{dU}{dx}\right\}dy$$

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As h →∞

$$\int_0^\infty (U-u)dy = U\delta^*$$

$$\int_0^\infty u(U-u)dy = U^2\theta$$

von Karman Momentum Integral

$$\frac{\tau_w}{\rho} = \frac{d(U^2\theta)}{dx} + \delta^* U \frac{dU}{dx}$$

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Karman – Pohlhausen Method

$$U^2\frac{d\theta}{dx} + (2\theta + \delta^*)U\frac{dU}{dx} = \frac{\tau_w}{\rho}$$

Multiplying by θ/νU

$$\frac{U\theta\frac{d\theta}{dx}}{\nu} + (2 + \frac{\delta^*}{\theta})\frac{\theta^2}{\nu}\frac{dU}{dx} = \frac{\tau_w\theta}{\mu U}$$

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Let

$$Z = \frac{\theta^2}{\nu}$$

$$K = \frac{\theta^2}{\nu}\frac{dU}{dx} = Z\frac{dU}{dx}$$

Then

$$U\frac{dZ}{dx} = F(K) = \frac{2\tau_w\theta}{\mu U} - 2(2 + \frac{\delta^*}{\theta})K$$

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Walz Approximation

$$F(K) \approx a - bK \qquad a = 0.47, \quad b = 6$$

Then

$$U\frac{dZ}{dx} = a - bK$$

Or

$$U\frac{dZ}{dx} = a - bZ\frac{dU}{dx}$$

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**Let**

$$Y = UZ = \frac{U\theta^2}{\nu}$$

**Then**

$$\frac{dY}{dx} + \frac{(b-1)}{U}\frac{dU}{dx}Y = a$$

**Solution, multiplying by $U^{b-1}$**

$$\frac{d}{dx}(YU^{b-1}) = aU^{b-1} \Rightarrow Y = \frac{a}{U^{b-1}(x)}\int_0^x U^{b-1}(x_1)dx_1$$

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**Therefore**

$$\frac{U\theta^2}{\nu} = \frac{0.47}{U^5(x)}\int_0^x U^5(x_1)dx_1$$

**Example**
**For a flat plate U= $U_o$**

$$\frac{U_o\theta^2}{\nu} = \frac{0.47xU_o^5}{U_o^5} \Rightarrow \theta = 0.686\sqrt{\frac{\nu x}{U_o}}$$

**Exact Solution**

$$\theta = 0.664\sqrt{\frac{\nu x}{U_o}}$$

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**Example**
**For U= $u_{o1}x^m$**

$$\theta = \sqrt{\frac{0.47}{5m+1}}\sqrt{\frac{\nu x}{U(x)}} = \sqrt{\frac{0.47}{5m+1}}\sqrt{\frac{\nu}{u_1 x^{m-1}}}$$

**m=0.5**

$$\theta = 0.336\sqrt{\frac{\nu x}{U(x)}} = 0.336\sqrt{\frac{\nu x^{1/2}}{u_1}}$$

**m=1**

$$\theta = 0.28\sqrt{\frac{\nu x}{U(x)}} = 0.28\sqrt{\frac{\nu}{u_1}}$$

**m=2**

$$\theta = 0.207\sqrt{\frac{\nu x}{U(x)}} = 0.207\sqrt{\frac{\nu}{u_1 x}}$$

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## Concluding Remarks

- Momentum Integral Method
- Flat Plate Flows
- Boundary Layer with Pressure Gradient
- Momentum Integral Method with Pressure Gradient
- Walz Approximation

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