ME 527 – Advanced Fluids

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Viscous Flows at High Reynolds Number

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Outline

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- Plane Flows and Stream Function
- High Reynolds Number Flows
- Outer Solution
- Inner Solution
- Matching
- Boundary Layer Equation

Two-Dimensional Plane Flows

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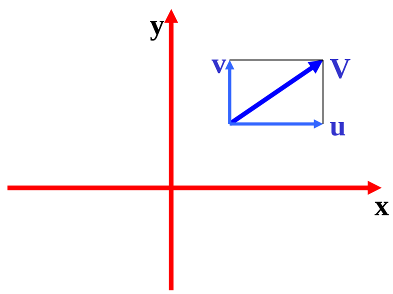
Stream Function

$$\mathbf{v} = \nabla \times (\mathbf{k}\psi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix} = \frac{\partial \psi}{\partial y}\mathbf{i} - \frac{\partial \psi}{\partial x}\mathbf{j}$$



$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

Two-Dimensional Plane Flows

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Vorticity Transport Equation in terms of Stream Function

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi = \frac{1}{\mathrm{Re}}\nabla^4\psi$$

Steady Flows

$$\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{1}{\mathrm{Re}}\nabla^2\right)\nabla^2\psi = 0$$

Two-Dimensional Plane Flows

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$$\left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}-\varepsilon^2\nabla^2\right)\nabla^2\psi=0,\quad \varepsilon^2=\frac{1}{\mathrm{Re}}$$

Boundary Conditions →

$$u=\frac{\partial \psi}{\partial y}=0,\ \psi=0 \quad at \ \ y=0$$

$$\psi=y \qquad as \ \ y\to\infty$$

Outer Expansion

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$$\psi=\psi_o+\varepsilon\psi_1+\ldots$$

Keeping terms ~ ε⁰

$$\left(\frac{\partial \psi_o}{\partial y}\frac{\partial}{\partial x}-\frac{\partial \psi_o}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi_o=0$$

Integrating for irrotational incoming flow

$$\nabla^2\psi_o=-\omega_o(\psi_o)=0$$

Boundary Conditions

$$\psi=0 \quad at \ \ y=0$$

$$\psi=y \quad as \ \ y\to\infty$$

→ $$\psi_o=y$$

Inner Expansion

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$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}-\varepsilon^2\nabla^2\right)\nabla^2\psi=0$$

Inner Variables

$$u=U_o(x,Y),\quad v=\varepsilon V_o(x,Y),\quad y=\varepsilon Y,\quad \psi=\varepsilon\Psi_o(x,Y)$$

$$\left[U_0\frac{\partial}{\partial x}+V_0\frac{\partial}{\partial Y}-\varepsilon^2\left(\frac{\partial^2}{\partial x^2}+\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial Y^2}\right)\right]\left(\frac{\partial^2}{\partial x^2}+\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial Y^2}\right)\varepsilon\Psi_0=0$$

Inner Expansion

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Hence

$$\left(U_0\frac{\partial}{\partial x}+V_0\frac{\partial}{\partial Y}-\frac{\partial^2}{\partial Y^2}\right)\frac{\partial^2\Psi_0}{\partial Y^2}=0$$

$$U_0\frac{\partial^2 U_0}{\partial x\partial Y}+V_0\frac{\partial^2 U_0}{\partial Y^2}-\frac{\partial^3 U_0}{\partial Y^3}=0,\quad U_0=\frac{\partial \Psi_0}{\partial Y}$$

Therefore

$$\frac{\partial}{\partial Y}\left(U_0\frac{\partial U_0}{\partial x}+V_0\frac{\partial U_0}{\partial Y}-\frac{\partial^2 U_0}{\partial Y^2}\right)=0$$

Inner Expansion

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Since

$$\left(\frac{\partial U_0}{\partial Y}\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial Y}\frac{\partial U_0}{\partial Y}\right)=\frac{\partial U_0}{\partial Y}\left(\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial Y}\right)=0$$

Therefore, the inner solution is given as

$$U_0\frac{\partial U_0}{\partial x}+V_0\frac{\partial U_0}{\partial Y}-\frac{\partial^2 U_0}{\partial Y^2}=f(x)$$

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Matching

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The outer limit of (inner expansion) = The inner limit of (outer expansion)

The inner limit of (outer expansion)

$$\lim_{\varepsilon\to 0} u_o(x,\varepsilon Y)=u_o(x,0)$$

The outer limit of (inner expansion)

$$\lim_{\varepsilon\to 0} U_o(x,y/\varepsilon)=U_o(x,\infty)$$

Therefore

$$u_o(x,0)=U_o(x,\infty)$$

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Inner Expansion

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For the inner solution to be valid

$$y\to\infty$$

$$u_0(x,0)\frac{du_0(x,0)}{dx}=f(x)=-\frac{dp_o}{dx}$$

Therefore, the inner solution is given as

$$U_0\frac{\partial U_0}{\partial x}+V_0\frac{\partial U_0}{\partial Y}=u_0(x,0)\frac{du_0(x,0)}{dx}+\frac{\partial^2 U_0}{\partial Y^2}$$

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Boundary Layer Equation

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$$U_0\frac{\partial U_0}{\partial x}+V_0\frac{\partial U_0}{\partial Y}=u_0\frac{du_0}{dx}+\frac{\partial^2 U_0}{\partial Y^2}$$

Or

$$U_0\frac{\partial U_0}{\partial x}+V_0\frac{\partial U_0}{\partial Y}=-\frac{dp_0}{dx}+\frac{\partial^2 U_0}{\partial Y^2}$$

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Concluding Remarks

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Thank you!

Questions?