ME 527 – Advanced Fluids

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Incompressible Viscous Flows

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Outline

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- Navier-Stokes Equation
- Vorticity Transport Equation
- Plane Flows and Stream Function
- Cylindrical Flows
- Spherical Coordinate Systems
- Plane Stagnation Flows
- Axisymmetric Stagnation Flows

Incompressible Viscous Flows

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Continuity

$$\nabla \cdot \mathbf{v} = 0$$

Navier-Stokes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v}$$

Incompressible Viscous Flows

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Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}$$

Vorticity Transport Equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = -\nu \nabla \times \nabla \times \boldsymbol{\omega}$$

Vorticity Transport Equation

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Vector Identity

$$\nabla\times(\mathbf{v}\times\boldsymbol{\omega}) = (\nabla\cdot\boldsymbol{\omega})\mathbf{v} - (\nabla\cdot\mathbf{v})\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\boldsymbol{\omega}$$

Vorticity Transport Equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \mathbf{v}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}$$

Two-Dimensional Plane Flows

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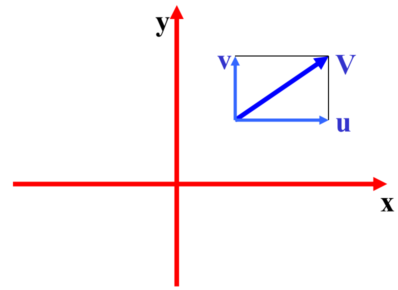
Stream Function

$$\mathbf{v} = \nabla\times(\mathbf{k}\psi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix} = \frac{\partial\psi}{\partial y}\mathbf{i} - \frac{\partial\psi}{\partial x}\mathbf{j}$$



$$u = \frac{\partial\psi}{\partial y}$$

$$v = -\frac{\partial\psi}{\partial x}$$

Two-Dimensional Plane Flows

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Vorticity

$$\omega_Z = \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2\psi$$

Vorticity Transport Equation

$$\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = \nu\nabla^2\omega$$

Two-Dimensional Plane Flows

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Vorticity Transport Equation in terms of Stream Function

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi = \nu\nabla^4\psi$$

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial(\nabla^2\psi,\psi)}{\partial(x,y)} = \nu\nabla^4\psi$$

Plane Cylindrical Flows

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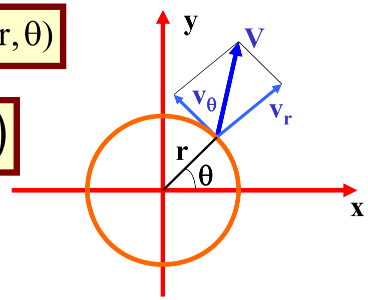
$v_z = 0$ $v_r(r,\theta)$ $v_\theta(r,\theta)$

$$\mathbf{v} = \nabla \times \left(\mathbf{e}_z \psi(r,\theta)\right)$$

$$\mathbf{v} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\mathbf{e}_r - \frac{\partial \psi}{\partial r}\mathbf{e}_\theta$$



$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

Plane Cylindrical Flows

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Vorticity → $\omega_z = \omega = -\nabla^2 \psi$

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}$$

Vorticity Transport Equation

$$\frac{\partial \omega}{\partial t} + v_r \frac{\partial \omega}{\partial r} + v_\theta \frac{1}{r}\frac{\partial \omega}{\partial \theta} = \nu \nabla^2 \omega$$

Plane Cylindrical Flows

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Stream Function Equation

$$\frac{\partial}{\partial t}\nabla^2 \psi + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r}\nabla^2 \psi - \frac{\partial \psi}{\partial r}\frac{1}{r}\frac{\partial}{\partial \theta}\nabla^2 \psi = \nu \nabla^4 \psi$$

⇕

$$\frac{\partial}{\partial t}\nabla^2 \psi + \frac{1}{r}\frac{\partial\left(\nabla^2 \psi, \psi\right)}{\partial (r,\theta)} = \nu \nabla^4 \psi$$

Axisymmetric Flows

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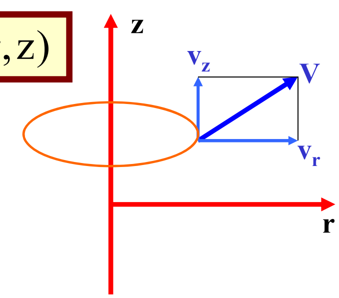
$v_\theta = 0$ $v_r(r,z)$ $v_z(r,z)$

$$\mathbf{v} = \nabla \times \left(\mathbf{e}_\theta \frac{\psi}{r}(r,z)\right)$$

$$\mathbf{v} = -\frac{1}{r}\frac{\partial \psi}{\partial z}\mathbf{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial r}\mathbf{e}_z$$



$$v_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \qquad v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$$

Axisymmetric Flows

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Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = \mathbf{e}_\theta \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right)$$

$$\boldsymbol{\omega} = -\mathbf{e}_\theta \frac{1}{r} \left(\frac{\partial^2 \psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right)$$

$$\omega_\theta = \omega = -\frac{1}{r} E^2 \psi$$

$$E^2 \psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}$$

Axisymmetric Flows

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Vorticity Transport Equation

$$\frac{\partial \omega_\theta}{\partial t} + v_r \frac{\partial \omega_\theta}{\partial r} + v_z \frac{\partial \omega_\theta}{\partial z} - \frac{v_r \omega_\theta}{r} = -\nu \nabla \times \nabla \times (\omega_\theta \mathbf{e}_\theta)|_{\theta-\text{comp.}}$$

$$\frac{\partial}{\partial t}(E^2\psi) - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial}{\partial r}(E^2\psi) + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial}{\partial z}(E^2\psi) + \frac{2}{r^2}\frac{\partial \psi}{\partial z} E^2\psi = \nu E^4 \psi$$

$$\frac{\partial}{\partial t}(E^2\psi) - \frac{1}{r}\frac{\partial (E^2\psi, \psi)}{\partial (r, z)} + \frac{2}{r^2}\frac{\partial \psi}{\partial z} E^2\psi = \nu E^4 \psi$$

Spherical Coordinates

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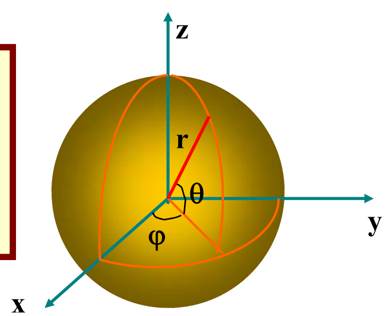
$$\begin{cases} x = r \cos\theta \cos\varphi \\ y = r \cos\theta \sin\varphi \\ z = r \sin\theta \end{cases}$$



Spherical Coordinates

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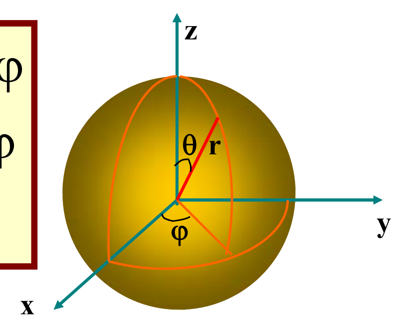
$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$



Spherical Coordinates

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$v_\varphi = 0$ $v_r(r,\theta)$ $v_\theta(r,\theta)$

Stream Function

$$\mathbf{v} = \nabla \times \left(\frac{\mathbf{e}_\varphi \psi(r,\theta)}{r\sin\theta} \right)$$

$$\mathbf{v} = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}\mathbf{e}_r - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\mathbf{e}_\theta$$

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \qquad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

Spherical Coordinates

Clarkson University

Vorticity Transport Equation

$$\frac{\partial}{\partial t}\left(E^2\psi\right) + \frac{1}{r^2\sin\theta}\frac{\partial\left(E^2\psi,\psi\right)}{\partial(r,\theta)} + \frac{2E^2\psi}{r^2\sin^2\theta}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right) = \nu E^4\psi$$

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}$$

Intrinsic Coordinates

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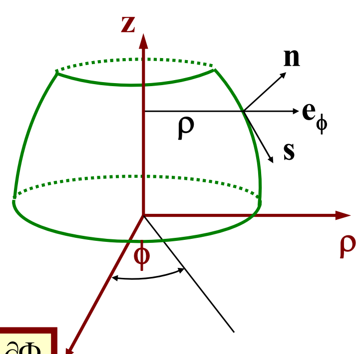
Metric Coefficients

$h_1 = h_n = 1$

$h_2 = h_\varphi = \rho$

$h_3 = h_s = 1$



$$\nabla\Phi = \mathbf{n}\frac{\partial\Phi}{\partial n} + \mathbf{s}\frac{\partial\Phi}{\partial s} + \mathbf{e}_\varphi\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi}$$

Intrinsic Axisymmetric Flows

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Stream Function

$$\mathbf{v} = \nabla \times \left(\mathbf{e}_\varphi \frac{\psi(s,n)}{\rho} \right) = -\frac{1}{\rho}\frac{\partial\psi}{\partial s}\mathbf{n} + \frac{1}{\rho}\frac{\partial\psi}{\partial n}\mathbf{s}$$

$$v_n = -\frac{1}{\rho}\frac{\partial\psi}{\partial s} \qquad v_s = \frac{1}{\rho}\frac{\partial\psi}{\partial n}$$

Intrinsic Axisymmetric Flows

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Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = -\mathbf{e}_\varphi \left[\frac{\partial}{\partial n}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial n}\right) + \frac{\partial}{\partial s}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial s}\right)\right] = -\frac{1}{\rho}E^2\psi$$

$$E^2 = \rho\left[\frac{\partial}{\partial n}\left(\frac{1}{\rho}\frac{\partial}{\partial n}\right) + \frac{\partial}{\partial s}\left(\frac{1}{\rho}\frac{\partial}{\partial s}\right)\right]$$

$$E^2 = \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}$$

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Plane Stagnation Flows

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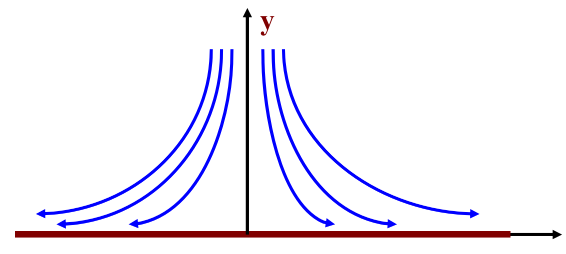
Navier-Stokes Equation

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi = \nu\nabla^4\psi$$



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Plane Stagnation Flows

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Potential Flow → $\psi = axy$; $U = ax$; $V = -ay$

ViscousFlow → $\psi = xf(y)$

$u = xf'$; $v = f$

$\nabla^2\psi = xf''$

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Navier-Stokes → $f'f'' - ff''' = \nu f^{(4)}$

Integrating → $f'^2 - ff'' = \nu f''' + c$

Boundary Conditions →

$y = 0$: $u = 0$, $v = 0$

$y \to \infty$: $\psi = axy$

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Plane Stagnation Flows

Boundary Conditions

$$f(0) = f'(0) = 0$$

$$y \to \infty \qquad f \to ay$$

Navier-Stokes

$$f'^2 - ff'' = \nu f''' + a^2$$

Plane Stagnation Flows

Changing Variables

$$\eta = \sqrt{\frac{a}{\nu}}\,y \qquad f = \sqrt{a\nu}\,\varphi(\eta)$$

Navier-Stokes

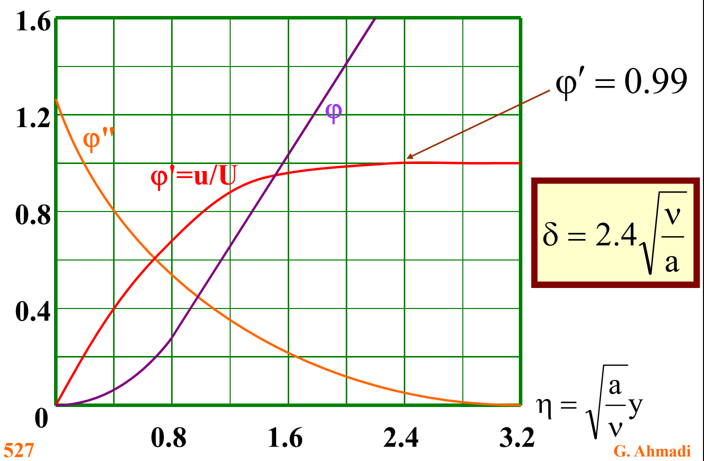
$$\varphi''' + \varphi\varphi'' - \varphi'^2 + 1 = 0$$

Boundary Conditions

$$\phi(0) = \phi'(0) = 0$$

$$\eta \to \infty \qquad \varphi' = 1$$

Plane Stagnation Flows



$$\delta = 2.4\sqrt{\frac{\nu}{a}}$$

Concluding Remarks

- **Navier-Stokes Equation for Viscous Flows**
- **Vorticity Transport Equation**
- **Plane Flows and Stream Function**
- **Cylindrical Spherical Coordinate Systems**
- **Stagnation Flows**