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Steady Parallel Viscous Flows

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Viscous Flows

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Outline

- Cartesian Coordinates
- Cylindrical Axial Flows
- Cylindrical Rotating Flows

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Cartesian Coordinates

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Incompressible Fluid

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

4 Equations for 4 unknowns u,v,w and p

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Cartesian Coordinates

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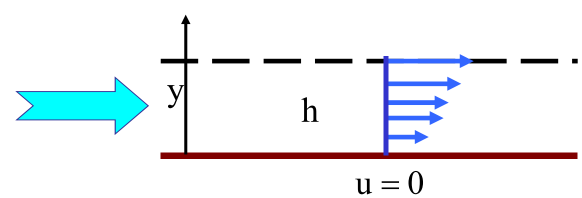

$$u(y),\ v=0,\ w=0$$

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Cartesian Coordinates

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Steady — Fully Developed — v=0 — w=0

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

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Cartesian Coordinates

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Steady Parallel Flows

$$\rho g_x-\frac{\partial p}{\partial x}+\mu\frac{d^2 u}{dy^2}=0$$

General Solution

$$u=-\frac{1}{2\mu}\left(\rho g_x-\frac{\partial p}{\partial x}\right)y^2+Ay+B$$

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Boundary Conditions

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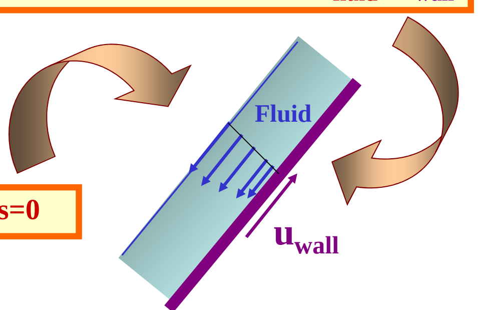
Solid Walls

A viscous fluid sticks to its boundary, $u_{fluid}=u_{wall}$

Free Surface

Shear Stress=0



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Boundary Conditions

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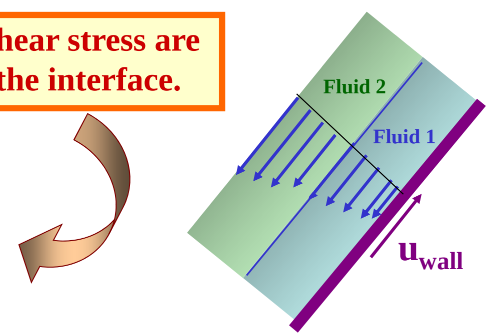
Between Two Fluids

Velocity and shear stress are the same at the interface.

$u_1 = u_2$

$\tau_1 = \tau_2$



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Cylindrical Coordinates

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r-Momentum

$$\rho\left(\frac{\partial v_r}{\partial t} + \mathbf{V}\cdot\nabla v_r - \frac{1}{r}v_\theta^2\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) + \rho g_r$$

θ-Momentum

$$\rho\left(\frac{\partial v_\theta}{\partial t} + \mathbf{V}\cdot\nabla v_\theta + \frac{1}{r}v_r v_\theta\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) + \rho g_\theta$$

z-Momentum

$$\rho\left(\frac{\partial v_z}{\partial t} + \mathbf{V}\cdot\nabla v_z\right) = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z + \rho g_z$$

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Cylindrical Coordinates

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Mass

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Convective

$$\mathbf{V}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{1}{r}v_\theta\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z}$$

Laplacian

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

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Cylindrical Coordinates

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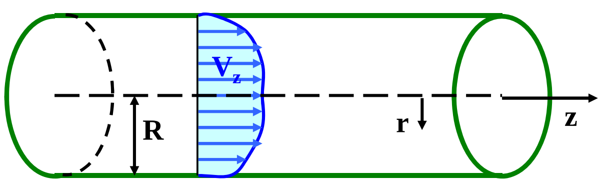
Steady Parallel Axial Flows

$$v_z(r),\ v_r = 0,\ v_\theta = 0$$



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Steady Parallel Axial Flows

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$v_r=0$ $v_\theta=0$ Fully Developed

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \equiv 0$$

$$\mathbf{V}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{1}{r}v_\theta\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z} \equiv 0$$

symmetry

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

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Steady Parallel Axial Flows

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Steady **$v_r=0$** **$v_\theta=0$**

$$\rho\left(\frac{\partial v_r}{\partial t} + \mathbf{V}\cdot\nabla v_r - \frac{1}{r}v_\theta^2\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) + \rho g_r$$

symmetry

$$\rho\left(\frac{\partial v_\theta}{\partial t} + \mathbf{V}\cdot\nabla v_\theta + \frac{1}{r}v_r v_\theta\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + \mathbf{V}\cdot\nabla v_z\right) = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z + \rho g_z$$

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Cylindrical Coordinates

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Steady Parallel Axial Flows

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = 0$$

General Solution

$$v_z = -\frac{1}{4\mu}\left(\rho g_z - \frac{\partial p}{\partial z}\right)r^2 + A\ln r + B$$

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Cylindrical Coordinates

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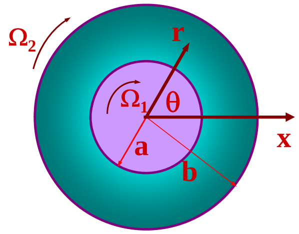
Steady Parallel Rotating Flows



$$v_z = 0,\ v_r = 0,\ v_\theta(r)$$

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Steady Parallel Rotating Flows

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$v_\theta(r)$ **$v_r=0$** **symmetry** **$v_z=0$**

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \equiv 0$$

$$\mathbf{V}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{1}{r}v_\theta\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z} \equiv 0$$

Two-D Flow

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

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Steady Parallel Rotating Flows

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Steady $v_r=0$ $v_z=0$ symmetry $v_\theta(r)$

$$\rho\left(\frac{\partial v_r}{\partial t} + \mathbf{V}\cdot\nabla v_r - \frac{1}{r}v_\theta^2\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + \mathbf{V}\cdot\nabla v_\theta + \frac{1}{r}v_r v_\theta\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + \mathbf{V}\cdot\nabla v_z\right) = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z + \rho g_z$$

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Cylindrical Coordinates

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Steady Parallel Rotating Flows

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right) - \frac{v_\theta}{r^2} = 0 \quad\Rightarrow\quad \frac{d^2 v_\theta}{dr^2} + \frac{1}{r}\frac{dv_\theta}{dr} - \frac{v_\theta}{r^2} = 0$$

General Solution

Euler's Equation

$$v_\theta = Ar + \frac{B}{r}$$

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Steady Parallel Rotating Flows

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r-Component

$$\frac{dp}{dr} = \rho\frac{v_\theta^2}{r}$$

$$\frac{dp}{dr} = \rho\frac{1}{r}\left(Ar + \frac{B}{r}\right)^2$$

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Summary

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Cartesian

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\frac{d^2 u}{dy^2} = 0 \qquad u = -\frac{1}{2\mu}\left(\rho g_x - \frac{\partial p}{\partial x}\right)y^2 + Ay + B$$

Cylindrical Axial

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = 0 \qquad v_z = -\frac{1}{4\mu}\left(\rho g_x - \frac{\partial p}{\partial z}\right)r^2 + A\ln r + B$$

Rotating

$$\frac{d^2 v_\theta}{dr^2} + \frac{1}{r}\frac{dv_\theta}{dr} - \frac{v_\theta}{r^2} = 0 \qquad v_\theta = Ar + \frac{B}{r} \qquad \frac{dp}{dr} = \rho\frac{v_\theta^2}{r}$$

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Steady Radial Flows

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Steady $v_\theta=0$ $v_z=0$ symmetry $v_r(r)$

$$\rho\left(\frac{\partial v_r}{\partial t}+\mathbf{V}\cdot\nabla v_r-\frac{1}{r}v_\theta^2\right)=-\frac{\partial p}{\partial r}+\mu\left(\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+\mathbf{V}\cdot\nabla v_\theta+\frac{1}{r}v_r v_\theta\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\nabla^2 v_\theta-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t}+\mathbf{V}\cdot\nabla v_z\right)=-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z+\rho g_z$$

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Steady Radial Flows

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$v_r(r)$

r - Momentum

$$\rho v_r\frac{dv_r}{dr}=-\frac{\partial p}{\partial r}+\mu\left(\frac{d^2v_r}{dr^2}+\frac{1}{r}\frac{dv_r}{dr}-\frac{v_r}{r^2}\right)$$

Mass

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}=0 \qquad v_r=\frac{A}{r}$$

$$\frac{dp}{dr}=\rho\frac{A^2}{r^3} \qquad p=p_o-\frac{\rho A^2}{2r^2}$$

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Steady Spiral Flows

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$v_r(r)$ $v_\theta(r)$

$$\rho\left(v_r\frac{dv_r}{dr}-\frac{1}{r}v_\theta^2\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{d^2v_r}{dr^2}+\frac{1}{r}\frac{dv_r}{dr}-\frac{v_r}{r^2}\right)$$

$$\rho\left(v_r\frac{dv_\theta}{dr}+\frac{1}{r}v_r v_\theta\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\frac{d^2v_\theta}{dr^2}+\frac{1}{r}\frac{dv_\theta}{dr}-\frac{v_\theta}{r^2}\right)$$

Solution

$$v_r=\frac{C}{r} \qquad v_\theta=\frac{K}{r}$$

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Slider Bearings

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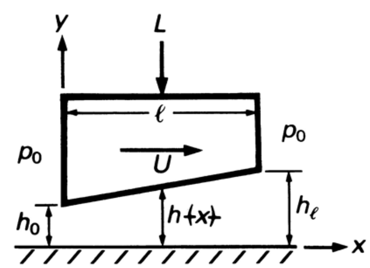
$$h_o/\ell<<1,\quad h_\ell/\ell<<1,\quad \frac{\partial u}{\partial x}\approx 0$$



$$-\frac{\partial p}{\partial x}+\mu\frac{d^2u}{dy^2}=0$$

$$u=\frac{1}{2\mu}\frac{\partial p}{\partial x}y^2+Ay+B$$

$$u=-\frac{1}{2\mu}\frac{\partial p}{\partial x}(hy-y^2)+\frac{Uy}{h}$$

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Viscous Flows

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Concluding Remarks

- Cartesian Coordinates
- Cylindrical Axial Flows
- Cylindrical Rotating Flows

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Thank you!

Questions?

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