CONSERVATION LAWS

Goodarz Ahmadi
Department of Mechanical and Aeronautical Engineering
Clarkson University
Potsdam, NY 13699-5725

CONSERVATION LAWS

Outline

- Conservation Laws
- Constitutive Equations
- Navier-Stokes Equation
- Heat Transfer Equation
- Dimensionless Groups

CONSERVATION LAWS

Axiom 1: Principle of Conservation of Mass

Mass is invariant under the motion

$$\frac{d}{dt}\int_{v}\rho dv = 0$$

Global

$$\frac{\partial}{\partial t}\int_{v}\rho dv + \int_{s}\rho \mathbf{v}\cdot \mathbf{ds} = 0$$

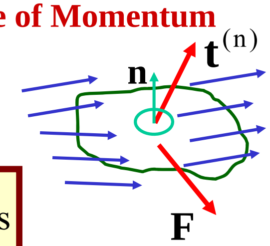
Local

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0$$

CONSERVATION LAWS

Axiom 2: Principle of Balance of Momentum

$$\frac{d}{dt}(\text{Momentum}) = \sum \text{Forces}$$



Global

$$\frac{d}{dt}\int_{v}\rho v_k dv = \int_{v}\rho f_k dv + \int_{s} t_k^{(n)} ds$$

Stress Tensor

$$\mathbf{t}^{(n)} = \mathbf{n}\cdot\mathbf{t}$$

Divergence Theorem

$$\int_{s} t_{\ell k} n_{\ell} ds = \int_{v} t_{\ell k,\ell} dv$$

CONSERVATION LAWS

Clarkson University

Axiom 2: Principle of Balance of Momentum

$$\int_v \left(\rho \frac{dv_k}{dt} - \rho f_k - t_{\ell k,\ell}\right) dv = 0$$

Local

$$\rho \frac{dv_k}{dt} = \rho f_k + t_{\ell k,\ell}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{t}$$

ME 527 G. Ahmadi

CONSERVATION LAWS

Clarkson University

Axiom 3: Principle of Balance of Angular Momentum

Global

$$\underbrace{\frac{d}{dt}\int_v \rho(\sigma_k + \varepsilon_{kmj} r_m v_j) dv}_{\text{Time rate of change of angular momentum}} = \underbrace{\int_v \rho \varepsilon_{kmj} r_m f_j dv}_{\text{Moment of body force}}$$

$$+ \underbrace{\int_S \varepsilon_{kmj} r_m t_j^{(n)} ds}_{\text{Moment of surface force}} + \int_s \underbrace{m_k^{(n)} ds}_{\text{Couple Stress}} + \underbrace{\int_s \rho \ell_k ds}_{\text{Body couple}}$$

ME 527 G. Ahmadi

Axiom 3: Principle of Balance of Angular Momentum

Clarkson University

Local

$$\rho \dot{\sigma}_k = \rho \ell_k + \varepsilon_{kmj} t_{mj} + m_{\ell k,\ell}$$

When $\sigma_k = \ell_k = m_{k\ell} = 0$ $\quad \varepsilon_{kmj} t_{mj} = 0$

$$t_{mj} = t_{jm}$$

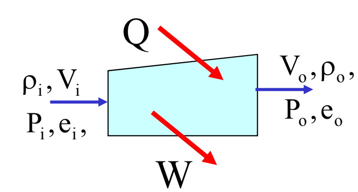
⟷ **Stress Tensor is Symmetric**

ME 527 G. Ahmadi

CONSERVATION LAWS

Clarkson University

Axiom 4: Principle of Conservation of Energy



$$\underbrace{\frac{d}{dt}(K + E)}_{\substack{\text{Time rate of change} \\ \text{kinetic and internal} \\ \text{enrgies}}} = \underbrace{W}_{\substack{\text{Work done} \\ \text{by all the forces}}} + \underbrace{Q}_{\text{Heat transferred}}$$

ME 527 G. Ahmadi

Axiom 4: Principle of Conservation of Energy

Clarkson University

Global

$$\frac{d}{dt}\int_V \rho\left(e+\frac{1}{2}v_k v_k\right)dv=\int_V \rho v_k f_k dv + \int_S v_k \cdot t_k^{(n)} ds + \int_S q_k ds_k + \int_V \rho r dv$$

ME 527 G. Ahmadi

Axiom 4: Principle of Conservation of Energy

Clarkson University

Local

$$\rho\dot{e} = t_{\ell k} v_{\ell,k} + q_{k,k} + \rho r$$

q = heat flux
r = heat source

ME 527 G. Ahmadi

CONSERVATION LAWS

Clarkson University

Axiom 5: Entropy Inequality

Global

$$\underbrace{\frac{d}{dt}\int_V \rho\eta dv}_{\text{Time rate of change of entropy}} \underbrace{-\int_S \frac{q_k n_k}{T} ds - \int_V \frac{\rho r}{T} dv}_{\text{Heat transfer divided by temperature}} \ge 0$$

Local

$$\rho\dot{\eta} - \left(\frac{q_k}{T}\right)_{,k} - \frac{\rho r}{T} \ge 0$$

ME 527 G. Ahmadi

Summary of Conservation Laws - Local

Clarkson University

Mass $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$

Momentum $\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{f} + \nabla\cdot\mathbf{t}$ $\quad \mathbf{t} = \mathbf{t}^T$

Energy $\rho\frac{de}{dt} = \mathbf{t}:\nabla\mathbf{v} + \nabla\cdot\mathbf{q} + \rho r$

Entropy $\rho\frac{d\eta}{dt} - \nabla\cdot\left(\frac{\mathbf{q}}{T}\right) - \frac{\rho r}{T} \ge 0$

ME 527 G. Ahmadi

CONSTITUTIVE EQUATIONS

Clarkson University

Continuum Thermodynamics

Helmholtz Free Energy

$$\psi = e - T\eta$$

Entropy

$$\frac{\rho}{T}(\dot{e} - \dot{T}\eta - \dot{\psi}) - (\frac{q_k}{T})_{,k} - \frac{\rho r}{T} \ge 0$$

Clausius-Duhem

$$\frac{1}{T}\left[-\rho(\dot{\psi} + \eta\dot{T}) + t_{\ell k} v_{\ell,k} + \frac{q_k T_{,k}}{T}\right] \ge 0$$

ME 527 G. Ahmadi

CONSTITUTIVE EQUATIONS

Clarkson University

Constitutive Postulates

Assuming

$$\psi = \psi(T, \rho, d_{k\ell}, T_{,k})$$

$$\dot{\psi} = \frac{\partial \psi}{\partial T}\dot{T} + \frac{\partial \psi}{\partial \rho}\dot{\rho} + \frac{\partial \psi}{\partial d_{k\ell}}\dot{d}_{k\ell} + \frac{\partial \psi}{\partial T_{,k}}\dot{\overline{T_{,k}}}$$

Pressure

$$p = -\frac{\partial \psi}{\partial \rho^{-1}} = \rho^2 \frac{\partial \psi}{\partial \rho}$$

$$\dot{\rho} = -\rho d_{kk} \qquad \frac{\partial \psi}{\partial \rho}\dot{\rho} = -\frac{p}{\rho} d_{kk}$$

ME 527 G. Ahmadi

CONSTITUTIVE EQUATIONS

Clarkson University

Entropy Equation

$$\frac{1}{T}\left[-\rho(\frac{\partial \psi}{\partial T} + \eta)\dot{T} + (t_{k\ell} + p\delta_{k\ell})d_{k\ell} - \rho\frac{\partial \psi}{\partial T_{,k}}\dot{\overline{T_{,k}}} - \rho\frac{\partial \psi}{\partial d_{k\ell}}\dot{d}_{k\ell} + \frac{q_k T_{,k}}{T}\right] \ge 0$$

$$\eta = -\frac{\partial \psi}{\partial T} \qquad \frac{\partial \psi}{\partial T_{,k}} = \frac{\partial \psi}{\partial d_{k\ell}} = 0$$

$$(t_{k\ell} + p\delta_{k\ell})d_{k\ell} + \frac{q_k T_{,k}}{T} \ge 0$$

ME 527 G. Ahmadi

CONSTITUTIVE EQUATIONS

Clarkson University

Linear Constitutive Equations

$$t_{k\ell} = -p\delta_{k\ell} + L_{k\ell ij} d_{ij}$$

$$q_k = L_{kj} T_{,j}$$

$$L_{k\ell ij} d_{ij} d_{k\ell} \ge 0 \qquad L_{kj} T_{,k} T_{,j} \ge 0$$

ME 527 G. Ahmadi

Linear Constitutive Equations

Clarkson University

Isotropic Materials

$$L_{k\ell ij} = \lambda\delta_{k\ell}\delta_{ij} + \mu(\delta_{ki}\delta_{\ell j} + \delta_{kj}\delta_{\ell i}) + \mu_1(\delta_{ki}\delta_{\ell j} - \delta_{kj}\delta_{\ell i})$$

$$L_{k\ell} = \kappa\delta_{k\ell}$$

Newtonian Fluids

$$t_{k\ell} = (-p + \lambda d_{ii})\delta_{k\ell} + 2\mu d_{k\ell}$$

Fourier Law

$$q_k = \kappa T_{,k}$$

ME 527 G. Ahmadi

Linear Constitutive Equations

Clarkson University

Thermodynamical Constraints

$$3\lambda + 2\mu \ge 0 \qquad \mu \ge 0 \qquad \kappa \ge 0$$

Stokes Assumption

$$\lambda = -\frac{2}{3}\mu \quad \Longleftrightarrow \quad p = -\frac{1}{3}t_{kk}$$

$$t_{k\ell} = -p\delta_{k\ell} + 2\mu d^D_{k\ell}$$

$$d^D_{ij} = d_{ij} - \frac{1}{3}d_{kk}\delta_{ij}$$

ME 527 G. Ahmadi

Navier-Stokes Equation

Clarkson University

$$\rho\frac{dv_k}{dt} = -p_{,k} + \mu v_{k,jj} + (\lambda + \mu)v_{j,jk} + \rho f_k$$

Incompressible Fluids

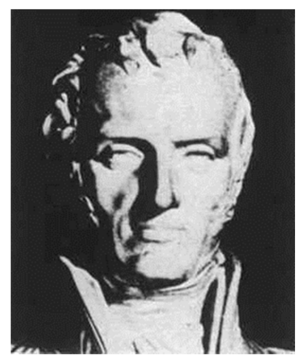
$$\nabla \cdot \mathbf{v} = 0$$

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{f}$$

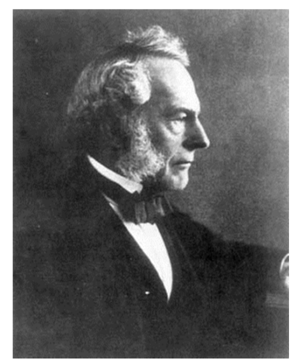
ME 527 G. Ahmadi

Navier-Stokes Equation

Clarkson University



Navier



Stokes

ME 527 G. Ahmadi

Energy Equation

Clarkson University

$$\rho \frac{de}{dt} = \mathbf{t} : \nabla \mathbf{v} + \nabla \cdot \mathbf{q} + \rho r$$

$$q_k = \kappa T_{,k}$$

Heat Transfer

$$\rho \frac{de}{dt} = \nabla \cdot (\kappa \nabla T) + t_{ij} v_{j,i} + \rho r$$

$$t_{ij} v_{j,i} = -p v_{k,k} + \Phi$$

Dissipation

$$\Phi = \lambda v_{k,k} v_{i,i} + 2\mu d_{ij} v_{j,i}$$

ME 527 G. Ahmadi

Energy Equation in term of Enthalpy

Clarkson University

$$p v_{k,k} = -\frac{p}{\rho}\frac{d\rho}{dt} = \rho \frac{d}{dt}\left(\frac{p}{\rho}\right) - \frac{dp}{dt}$$

Enthalpy

$$h = e + \frac{p}{\rho}$$

$$\rho \frac{dh}{dt} = \frac{dp}{dt} + \nabla \cdot (\kappa \nabla T) + \Phi + \rho r$$

Heat Capacities

$$dh = c_p dT$$

$$de = c_v dT$$

ME 527 G. Ahmadi

Heat Transfer Equation

Clarkson University

$$\rho c_P \frac{dT}{dt} = \frac{dP}{dt} + \kappa \nabla^2 T + \Phi + \rho r$$

Incompressible Flow

$$\rho c_v \frac{dT}{dt} = \kappa \nabla^2 T + \Phi + \rho r$$

Dissipation

$$\Phi = \mu (v_{i,j} + v_{j,i}) v_{j,i}$$

ME 527 G. Ahmadi

Boussinesq Approximation

Clarkson University

Thermal Expansion

$$\rho = \rho_0 (1 - \beta (T - T_0))$$

Body Force

$$\rho \mathbf{f} = -\rho_0 g \mathbf{k} \left[1 - \beta (T - T_0)\right]$$

Boussinesq Equation

$$\rho_0 \frac{d\mathbf{v}}{dt} = -\nabla \hat{P} + \mu \nabla^2 \mathbf{v} - \rho_0 g \beta (T - T_0) \mathbf{k}$$

$$\hat{P} = p + \rho_0 g z$$

ME 527 G. Ahmadi

Dimensionless Equations

Clarkson University

Dimensionless Variables

$$x_i^* = \frac{x_i}{L} \quad \mathbf{v}^* = \frac{\mathbf{v}}{U_\infty} \quad t^* = \frac{tU_\infty}{L} \quad \rho^* = \frac{\rho}{\rho_0}$$

$$P^* = \frac{\hat{P} - P_\infty}{\rho_0 U_\infty^2} \quad T^* = \frac{T - T_0}{\Delta T_0} \quad \mathbf{f}^* = \frac{\mathbf{f}}{g}$$

ME 527 G. Ahmadi

Dimensionless Equations

Clarkson University

Mass

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \mathbf{v}^*) = 0$$

Momentum

$$\rho^* \frac{d\mathbf{v}^*}{dt^*} = -\nabla^* P^* + \frac{1}{Re} \nabla^{*2} \mathbf{v}^* - \frac{Gr}{Re^2} T^* \mathbf{f}^*$$

Energy

$$\rho^* \frac{dT^*}{dt^*} = Ec \frac{dP^*}{dt^*} + \frac{1}{Re\,Pr} \nabla^{*2} T^* + \frac{Ec}{Re} \Phi^*$$

ME 527 G. Ahmadi

Dimensionless Groups

Clarkson University

Reynolds Number

$$Re = \frac{\rho_0 U_\infty L}{\mu}$$

Prandtl Number

$$Pr = \frac{\mu c_P}{\kappa}$$

Grashof Number

$$Gr = \frac{g \beta \rho_0^2 L^3 \Delta T_0}{\mu^2}$$

Eckert Number

$$Ec = \frac{U_\infty^2}{c_P \Delta T_0}$$

ME 527 G. Ahmadi

CONSERVATION LAWS

Clarkson University

Concluding Remarks

- Conservation Laws
- Constitutive Equations
- Navier-Stokes Equation
- Heat Transfer Equation
- Dimensionless Groups

ME 527 G. Ahmadi

Clarkson University

Thank you!

Questions?

ME 527 G. Ahmadi