ME 527 – Advanced Fluid

Clarkson University

Continuum Mechanics - Kinematics

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Kinematics

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Outline

- Motion & Inverse Motion
- Time Derivatives
- Velocity and Acceleration
- Deformation Rate Tensor
- Spin Tensor & Vorticity

Kinematics

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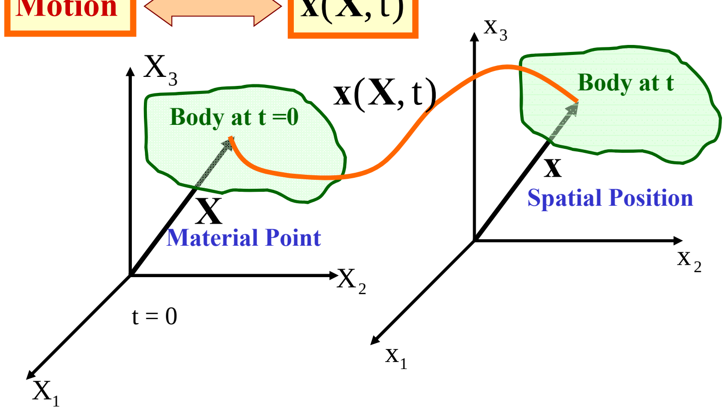
Motion ⟺ $\mathbf{x}(\mathbf{X}, t)$



Motion

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Body = Collection of Material Particles

X= Material Point = Position of particle at time zero

Motion: $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$

Inverse Motion: $\mathbf{X}=\mathbf{X}(\mathbf{x},t)$

Jacobian

$$J = \det\left|\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right| = \det\left|\frac{\partial x_k}{\partial X_K}\right| \neq 0$$

Streamlines

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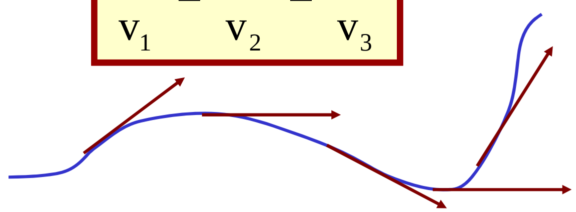
Streamlines are curves that are tangent to the velocity vector field

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}$$



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Streak Lines

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The streak line of point x^o at time t is a line, which is made up of material points, that have passed through point x^o at different times $\tau \le t$

X_k^0 passes through $\mathbf{x}^0$ at time τ

$$X_k^0 = X_k^0(\mathbf{x}^0, \tau)$$

Streak lines

$$x_i = x_i(\mathbf{X}^0(\mathbf{x}^0, \tau), t)$$

For fixed t

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Deformation Gradient

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$$dx_k = \frac{\partial x_k}{\partial X_K} dX_K = x_{k,K} dX_K$$

$$x_{k,K} = \frac{\partial x_k}{\partial X_K}$$

Deformation Gradient

$$X_{K,k} = \frac{\partial X_K}{\partial x_k}$$

Inverse Deformation Gradient

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Deformation Tensors

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Element of Arc in the Deformed Body

$$ds^2 = dx_k dx_\ell \delta_{k\ell}$$

Element of Arc in the Undeformed Body

$$dS^2 = dX_K X_L \delta_{KL}$$

Green Deformation Tensor

$$C_{KL} = \delta_{kl} x_{k,K} x_{\ell,L}$$

$$ds^2 = C_{KL} dX_K X_L$$

Cauchy Deformation Tensor

$$c_{kl} = X_{K,k} X_{L,\ell} \delta_{KL}$$

$$dS^2 = c_{k\ell} dx_k dx_\ell$$

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Strain Tensors

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Lagrangian Strain Tensor → $2E_{KL} = C_{KL} - \delta_{KL}$

$$ds^2 - dS^2 = 2E_{KL}dX_K dX_L$$

Eulerian Strain Tensor → $2e_{k\ell} = \delta_{k\ell} - c_{k\ell}$

$$ds^2 - dS^2 = 2e_{k\ell}dx_k dx_\ell$$

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Partial and Total Time Derivatives

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Partial Time Derivatives → $\frac{\partial A}{\partial t} = \left.\frac{\partial A}{\partial t}\right|_{\mathbf{x}}$

Material Time Derivatives → $\frac{dA}{dt} = \left.\frac{\partial A}{\partial t}\right|_{\mathbf{X}} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x_i}\left.\frac{\partial x_i}{\partial t}\right|_{\mathbf{X}}$

Velocity → $v_i = \left.\frac{\partial x_i}{\partial t}\right|_{\mathbf{X}} = \frac{dx_i}{dt} = \dot{x}_i$

Acceleration → $a_i = \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}$

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Path Line

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$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ → **For Fixed X**

$\frac{dx_i}{dt} = v_i(\mathbf{x}, t)$ → **Path Line**

Time Derivatives

$$\frac{d}{dt}(dx_k) = \frac{d}{dt}(x_{k,K}dX_K) = v_{k,K}dX_K = v_{k,\ell}dx_\ell$$

$$\frac{d}{dt}(x_{k,K}) = \frac{\partial}{\partial X_K}\frac{dx_k}{dt} = v_{k,K} = v_{k,\ell}x_{\ell,K}$$

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Deformation Rate Tensor

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$$d_{k\ell} = \frac{1}{2}(v_{k,\ell} + v_{\ell,k})$$

$$\frac{d}{dt}(ds^2) = 2d_{k\ell}dx_k dx_\ell$$

Identities

$$\dot{C}_{KL} = 2\dot{E}_{KL} = 2d_{k\ell}x_{k,K}x_{\ell,L}$$

$$2d_{k\ell} = \dot{C}_{KL}X_{K,k}X_{L,\ell} = 2\dot{E}_{KL}X_{K,k}X_{L,\ell}$$

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Rivlin-Ericksen Tensors

$$A^{(n+1)}_{k\ell} = \frac{d}{dt}A^{(n)}_{k\ell} + A^{(n)}_{km} v_{m,\ell} + A^{(n)}_{\ell m} v_{m,k}$$

$$A^{(1)}_{k\ell} = 2d_{k\ell}$$

$$A^{(2)}_{k\ell} = 2\dot{d}_{k\ell} + 2d_{km} v_{m,\ell} + 2d_{\ell m} v_{m,k}$$

$$\frac{d^n}{dt^n}(ds^2) = A^{(n)}_{k\ell}\, dx_k\, dx_\ell$$

Rate of Volume Change

$$\frac{d}{dt}J = J v_{k,k}$$

$$dv = J dV$$

$$\frac{d}{dt}dv = v_{k,k}\, dv$$

Rate of Volume Change

$$\det|\mathbf{A}| = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$$

$$J = \det\left|\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right| = \det\left|\frac{\partial x_k}{\partial X_K}\right| \neq 0$$

$$J = \varepsilon_{IJK} x_{1,I} x_{2,J} x_{3,K}$$

$$\frac{d}{dt}(x_{k,K}) = v_{k,\ell} x_{\ell,K}$$

$$\frac{dJ}{dt} = \varepsilon_{IJK} \frac{dx_{1,I}}{dt} x_{2,J} x_{3,K} + \ldots\ldots.$$

$$\frac{dJ}{dt} = \varepsilon_{IJK} v_{1,k} x_{k,I} x_{2,J} x_{3,K} + \ldots\ldots.$$

$$\frac{dJ}{dt} = \varepsilon_{IJK} v_{1,1} x_{1,I} x_{2,J} x_{3,K} + \ldots\ldots.$$

$$\frac{d}{dt}J = J v_{k,k}$$

Reynolds Transport Theorem

$$\frac{d}{dt}\iiint_v f\,dv = \iiint_v \frac{\partial f}{\partial t} dv + \oiint_s f\mathbf{v}\cdot\mathbf{ds}$$

Proof

$$\frac{d}{dt}\iiint_v f\,dv = \frac{d}{dt}\iiint_V f\,J\,dV = \iiint_V \left(\frac{df}{dt}J + f\frac{dJ}{dt}\right)dV$$

$$\frac{d}{dt}\iiint_v f\,dv = \iiint_V \left(\frac{df}{dt} + v_{k,k} f\right)J\,dV = \iiint_v (\dot{f} + v_{k,k} f)\,dv$$

$$\frac{d}{dt}\iiint_v f\,dv = \iiint_v \left(\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_k}(v_k f)\right)dv$$

Spin and Vorticity

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Spin Tensors → $\omega_{k\ell} = \frac{1}{2}(v_{k,\ell} - v_{\ell,k})$

Vorticity Vector: $\zeta_i = \varepsilon_{ijk}\omega_{kj} = \varepsilon_{ijk} v_{k,j}$

Angular Velocity Vector: $\omega_i = \frac{1}{2}\zeta_i$ $\quad \boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v}$

$(\nabla \mathbf{v})^T = \mathbf{d} + \boldsymbol{\omega}$ $\quad \zeta = \nabla \times \mathbf{v}$

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Concluding Remarks

- Motion & Inverse Motion
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Thank you!

Questions?

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