Advanced Fluid Mechanics

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Review of Engineering Mathematics

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Outline

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- Special Functions
- Differential Equations
- Fourier Series and Transforms
- Probability and Random Processes
- Linear System Analysis
- Vector Analysis

Special Functions

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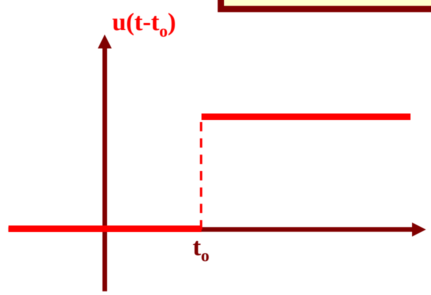
Unit Step Function

$$u(t-t_0)=\begin{cases}1 & t\ge t_0\\ 0 & t<t_0\end{cases}$$



Special Functions

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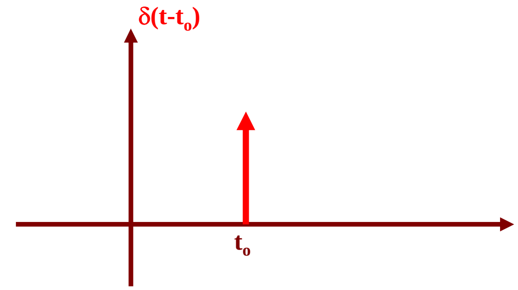
Dirac Delta Function

$$\delta(t-t_0)=\frac{du(t-t_0)}{dt}$$



Properties of Delta Function

$$\int_{-\infty}^{+\infty}\delta(t-t_0)dt=\int_{t_0-\varepsilon}^{t_0+\varepsilon}\delta(t-t_0)dt=1$$

$$\int_{-\infty}^{+\infty} f(t)\delta(t-t_0)dt=\int_{t_0-\varepsilon}^{t_0+\varepsilon} f(t)\delta(t-t_0)dt=f(t_0)$$

$$\int_{-\infty}^{t} f(t_1)\delta(t_1-t_0)dt_1=f(t_0)u(t-t_0)$$

$$\delta[a(t-t_0)]=\frac{1}{|a|}\delta(t-t_0)$$

Special Functions

Error Function

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

$$\operatorname{erfc}(x)=1-\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

$$\operatorname{erf}(0)=0=\operatorname{erfc}(\infty)$$

$$\operatorname{erf}(-x)=-\operatorname{erf}(x)$$

Exponential Integrals

$$E_i(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$$

$$E_n(x)=\int_1^\infty \frac{e^{-xt}}{t^n}dt$$

Differential Equations

Linear First-Order

$$\frac{dy}{dx}+P(x)y=Q(x)$$

$$y=ce^{-\int_0^x P(x_1)dx}+\int_0^x e^{-\int_{x_1}^x P(x_1)dx_2}Q(x_1)dx_1$$

$$\frac{dy}{dx}+b\,y=Q(x)$$

$$y=\int_0^x e^{-b(x-x_1)}Q(x_1)dx_1$$

Second-Order Equations

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$$

$$m^2+am+b=0 \quad \text{Solve for} \rightarrow \quad m_1, m_2$$

$$m_2\neq m_1=\text{Real}$$

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

$$m_2=m_1=m$$

$$y=e^{mx}(c_1+c_2x)$$

Second-Order Equations

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$m_1 = p + qi$

$m_2 = p - qi$

$p = -\frac{a}{2}$

$q = \sqrt{b - \frac{a^2}{4}}$

$$y = e^{px}\left(c_1 \cos qx + c_2 \sin qx\right)$$

Second-Order Equations

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Particular Solutions

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = R(x)$$

$$y_p = \frac{e^{m_1x}}{m_1 - m_2}\int e^{-m_1x}R(x)dx + \frac{e^{m_2x}}{m_2 - m_1}\int e^{-m_2x}R(x)dx$$

$$y_p = xe^{mx}\int e^{-mx}R(x)dx - e^{mx}\int xe^{-mx}R(x)dx$$

$$y_p = \frac{e^{px}\sin qx}{q}\int e^{-px}R(x)\cos qx\,dx - \frac{e^{px}\cos qx}{q}\int e^{-px}R(x)\sin qx\,dx$$

Euler Equation

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$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = s(x)$$

Let $y = Ax^m$ → $m(m-1) + am + b = 0$

$$y = A_1x^{m_1} + A_2x^{m_2}$$

Homogeneous Equations

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$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Let $v = \frac{y}{x}$ → $\frac{dy}{dx} = x\frac{dv}{dx} + v$

$$\ln x = \int \frac{dv}{F(v) - v} + c$$

Exact Equations

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$$M(x,y)dx + N(x,y)dy = 0$$

With

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 \varphi(x,y)}{\partial x \partial y}$$

$$M = \frac{\partial \varphi}{\partial x} \qquad N = \frac{\partial \varphi}{\partial y}$$

$$\varphi(x,y) = \text{const}$$

Bessel's Equation

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$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(\beta^2 x^2 - n^2\right)y = 0$$

Solutions

$$y = C_1 J_n(\beta x) + C_2 Y_n(\beta x)$$

Bessel Functions

$$J_n(\beta x) \qquad Y_n(\beta x)$$

Fourier Series

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Fourier Cosine Series

$$f(x) = f(-x)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

Fourier Sine Series

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$$f(x) = -f(-x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Fourier Exponential Series

$$\omega_n = \frac{n\pi}{L}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\omega_n x}dx$$

From Fourier Series to Fourier Transforms

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FES

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

$$-L < x < L$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-\frac{in\pi x}{L}} f(x')dx$$

Replacing for c_n

$$f(x) = \frac{1}{2L}\sum_{-\infty}^{+\infty}\int_{-L}^{L} e^{i\omega_n(x-x')} f(x')dx$$

$$\omega_n = \frac{n\pi}{L}$$

$$\Delta\omega = \frac{\pi}{L}$$

$$L \to \infty$$

$$\sum g_n \Delta\omega = \int g d\omega$$

Fourier Transforms

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Fourier Integral Representation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\omega(x-x')} f(x')dx'd\omega$$

Fourier Transform (Exponential)

$$\bar{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x'} f(x')dx'$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega x}\bar{f}(\omega)d\omega$$

Fourier Cosine and Sine Transforms

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$$\bar{f}_c(\omega) = \int_0^{\infty} \cos\omega x' f(x')dx'$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos\omega x \bar{f}_c(\omega)d\omega$$

Fourier Cosine Transform

$$\bar{f}_s(\omega) = \int_0^{\infty} \sin\omega x' f(x')dx'$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin\omega x \bar{f}_s(\omega)d\omega$$

Fourier Sine Transform

Fourier Transform of Derivatives

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$$\Im\left\{\frac{df}{dx}\right\} = \int_{-\infty}^{+\infty} e^{-i\omega x}\frac{df(x)}{dx}dx = i\omega\bar{f}(\omega)$$

$$\Im\left\{\frac{d^2 f}{dx^2}\right\} = -\omega^2\bar{f}(\omega)$$

$$\Im\left\{\frac{d^n f}{dx^n}\right\} = (i\omega)^n\bar{f}(\omega)$$

Example

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$$\frac{d^2 f}{dx^2} + a\frac{df}{dx} + bf = \delta(x - x_0)$$

$$-\infty < x < +\infty$$

Taking Fourier Transform

$$-\omega^2 \bar{f}(\omega) + ai\omega\bar{f}(\omega) + b\bar{f}(\omega) = e^{-i\omega x_0}$$

$$\bar{f}(\omega) = \frac{e^{-i\omega x_0}}{b - \omega^2 + ia\omega}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\omega(x - x_0)}}{b - \omega^2 + ia\omega} d\omega$$

Table of Fourier Transform

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$f(x)$	$\bar{f}(\omega)$
$f_1(x + x_0)$	$e^{i\omega x_0}\bar{f}(\omega)$
$\delta(x - x_0)$	$e^{-i\omega x_0}$
$e^{-\alpha\|x\|}$	$\frac{2\alpha}{\omega^2 + \alpha^2}$
$f_1(x) * f_2(x) = \int_{-\infty}^{+\infty} f_1(\xi) f_2(x - \xi) d\xi$	$\bar{f_1}(\omega)\bar{f_2}(\omega)$
$\cos\omega_0 x$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$e^{-\alpha\|x\|}\cos\beta x$	$\frac{2\alpha(\omega^2 + \alpha^2 + \beta^2)}{(\omega^2 - \beta^2 - \alpha^2)^2 + 4\alpha^2\omega^2}$
$e^{-\alpha\|x\|}\left[\cos\beta x + \frac{\alpha}{\beta}\sin\beta\|x\|\right]$	$\frac{4\alpha(\alpha^2 + \beta^2)}{(\omega^2 - \beta^2 - \alpha^2)^2 + 4\alpha^2\omega^2}$

Table of Fourier Transform

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$f(x)$	$\bar{f}(\omega)$
$e^{-\alpha^2 x^2}\cos\beta x$	$\frac{\sqrt{\pi}}{2\alpha}\left[\exp\left\{-\frac{(\omega+\beta)^2}{4\alpha^2}\right\} + \exp\left\{-\frac{(\omega-\beta)^2}{4\alpha^2}\right\}\right]$
$e^{-\alpha^2 x^2}$	$\frac{\sqrt{\pi}}{\alpha}\exp\left\{-\frac{\omega^2}{4\alpha^2}\right\}$
$\frac{d^n}{dx^n}\delta(x)$	$(i\omega)^n$
$J_0(x)$	$\begin{cases} \frac{2}{\sqrt{1-\omega^2}} & \|\omega\| < 1 \\ 0 & \text{elsewhere} \end{cases}$

Linear Systems Analysis – Impulse Response

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$$\dot{x} + \alpha x = f(t)$$

$$h(t) = e^{-\alpha t}$$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = f(t)$$

$$h(t) = \frac{1}{\omega_d} e^{-\zeta\omega_0 t}\sin\omega_d t$$

$$\omega_d = \omega_0\sqrt{1 - \zeta^2}$$

Vector Identities

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$$\nabla \cdot \nabla \times \vec{u} = 0$$

$$\nabla \times (\nabla \varphi) = 0$$

$$\nabla \times \nabla \times \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

$$\vec{u} \cdot \nabla \vec{u} = \nabla\left(\frac{u^2}{2}\right) - \vec{u} \times (\nabla \times \vec{u})$$

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Vector Identities

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$$\nabla \times (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \vec{u} - \vec{u} \cdot \nabla \vec{v} + (\nabla \cdot \vec{v})\vec{u} - (\nabla \cdot \vec{u})\vec{v}$$

$$\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \times \vec{u} - \vec{u} \cdot \nabla \times \vec{v}$$

$$\nabla(\vec{u} \cdot \vec{v}) = \vec{v} \cdot \nabla \vec{u} + \vec{u} \cdot \nabla \vec{v} + \vec{v} \times (\nabla \times \vec{u}) + \vec{u} \times (\nabla \times \vec{v})$$

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Thank you!

Questions?

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