ME 326 - Intermediate Fluid Mechanics

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Steady Parallel Viscous Flows

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Viscous Flows

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Outline

- Cartesian Coordinates
- Cylindrical Axial Flows
- Cylindrical Rotating Flows

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Cartesian Coordinates

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Incompressible Fluid

$$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})=\rho g_x-\frac{\partial p}{\partial x}+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})$$

$$\rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z})=\rho g_y-\frac{\partial p}{\partial y}+\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})$$

$$\rho(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z})=\rho g_z-\frac{\partial p}{\partial z}+\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

4 Equations for 4 unknowns u,v,w and p

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Cartesian Coordinates

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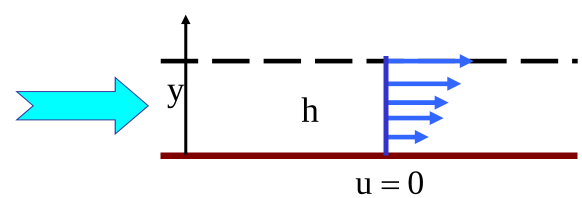
Steady Parallel Flows



$$u(y),\ v=0,\ w=0$$

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Steady Parallel Flows

Steady — Fully Developed — v=0 — w=0

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

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Cartesian Coordinates

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Steady Parallel Flows

$$\rho g_x-\frac{\partial p}{\partial x}+\mu\frac{d^2u}{dy^2}=0$$

General Solution

$$u=-\frac{1}{2\mu}\left(\rho g_x-\frac{\partial p}{\partial x}\right)y^2+Ay+B$$

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Boundary Conditions

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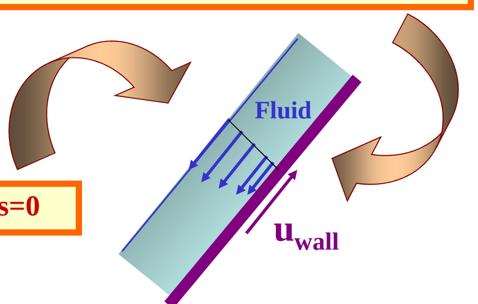
Solid Walls

A viscous fluid sticks to its boundary, $u_{fluid}=u_{wall}$

Free Surface

Shear Stress=0



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Boundary Conditions

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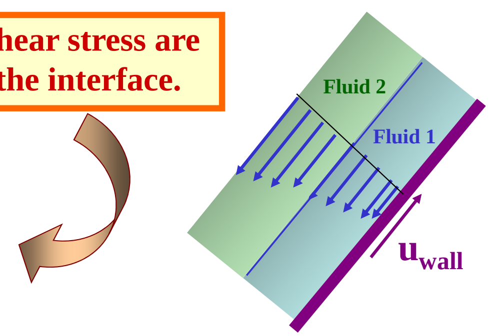
Between Two Fluids

Velocity and shear stress are the same at the interface.

$u_1 = u_2$

$\tau_1 = \tau_2$



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Cylindrical Coordinates

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r-Momentum

$$\rho\left(\frac{\partial v_r}{\partial t} + \mathbf{V}\cdot\nabla v_r - \frac{1}{r}v_\theta^2\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) + \rho g_r$$

θ-Momentum

$$\rho\left(\frac{\partial v_\theta}{\partial t} + \mathbf{V}\cdot\nabla v_\theta + \frac{1}{r}v_r v_\theta\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) + \rho g_\theta$$

z-Momentum

$$\rho\left(\frac{\partial v_z}{\partial t} + \mathbf{V}\cdot\nabla v_z\right) = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z + \rho g_z$$

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Cylindrical Coordinates

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Mass →

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Convective →

$$\mathbf{V}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{1}{r}v_\theta\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z}$$

Laplacian →

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

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Cylindrical Coordinates

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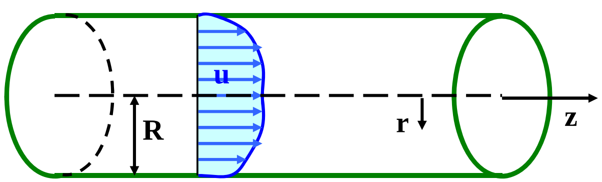
Steady Parallel Axial Flows

$$v_z(r),\ v_r = 0,\ v_\theta = 0$$



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Steady Parallel Axial Flows

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$v_r=0$ $v_\theta=0$ Fully Developed

$$\frac{1}{r}\frac{\partial(r \cancel{v_r})}{\partial r} + \frac{1}{r}\frac{\partial \cancel{v_\theta}}{\partial \theta} + \frac{\partial \cancel{v_z}}{\partial z} \equiv 0$$

$$\mathbf{V}\cdot\nabla = \cancel{v_r}\frac{\partial}{\partial r} + \frac{1}{r}\cancel{v_\theta}\frac{\partial}{\partial \theta} + v_z\frac{\cancel{\partial}}{\partial z} \equiv 0$$

symmetry

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2}{\partial \theta^2}} + \cancel{\frac{\partial^2}{\partial z^2}} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

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Steady Parallel Axial Flows

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Steady $v_r=0$ $v_\theta=0$

$$\rho\left(\frac{\partial v_r}{\partial t} + \mathbf{V}\cdot\nabla v_r - \frac{1}{r}v_\theta^2\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) + \rho g_r$$

symmetry

$$\rho\left(\frac{\partial v_\theta}{\partial t} + \mathbf{V}\cdot\nabla v_\theta + \frac{1}{r}v_r v_\theta\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + \mathbf{V}\cdot\nabla v_z\right) = -\frac{\partial p}{\partial z} + \mu\nabla^2 v_z + \rho g_z$$

Cylindrical Coordinates

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Steady Parallel Axial Flows

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = 0$$

General Solution

$$v_z = -\frac{1}{4\mu}\left(\rho g_z - \frac{\partial p}{\partial z}\right)r^2 + A\ln r + B$$

Cylindrical Coordinates

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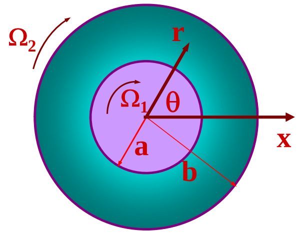
Steady Parallel Rotating Flows



$$v_z = 0,\ v_r = 0,\ v_\theta(r)$$

Steady Parallel Rotating Flows

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$v_\theta(r)$ $v_r=0$ symmetry $v_z=0$

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \equiv 0$$

$$\mathbf{V}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{1}{r}v_\theta\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z} \equiv 0$$

Two-D Flow

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

Steady Parallel Rotating Flows

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Steady v_r=0 v_z=0 symmetry $v_\theta(r)$

$$\rho\left(\frac{\partial v_r}{\partial t}+\mathbf{V}\cdot\nabla v_r-\frac{1}{r}v_\theta^2\right)=-\frac{\partial p}{\partial r}+\mu\left(\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+\mathbf{V}\cdot\nabla v_\theta+\frac{1}{r}v_r v_\theta\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\nabla^2 v_\theta-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t}+\mathbf{V}\cdot\nabla v_z\right)=-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z+\rho g_z$$

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Cylindrical Coordinates

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Steady Parallel Rotating Flows

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right)-\frac{v_\theta}{r^2}=0 \Rightarrow \frac{d^2v_\theta}{dr^2}+\frac{1}{r}\frac{dv_\theta}{dr}-\frac{v_\theta}{r^2}=0$$

General Solution

$$v_\theta=Ar+\frac{B}{r}$$

Euler's Equation

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Steady Parallel Rotating Flows

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r-Component

$$\frac{dp}{dr}=\rho\frac{v_\theta^2}{r}$$

$$\frac{dp}{dr}=\rho\frac{1}{r}\left(Ar+\frac{B}{r}\right)^2$$

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Summary

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Cartesian

$$\rho g_x-\frac{\partial p}{\partial x}+\mu\frac{d^2u}{dy^2}=0 \qquad u=-\frac{1}{2\mu}\left(\rho g_x-\frac{\partial p}{\partial x}\right)y^2+Ay+B$$

Cylindrical Axial

$$\rho g_z-\frac{\partial p}{\partial z}+\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=0 \qquad v_z=-\frac{1}{4\mu}\left(\rho g_x-\frac{\partial p}{\partial z}\right)r^2+A\ln r+B$$

Rotating

$$\frac{d^2v_\theta}{dr^2}+\frac{1}{r}\frac{dv_\theta}{dr}-\frac{v_\theta}{r^2}=0 \qquad v_\theta=Ar+\frac{B}{r} \qquad \frac{dp}{dr}=\rho\frac{v_\theta^2}{r}$$

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Slider Bearings

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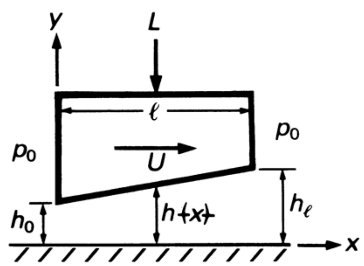
$$h_o/\ell << 1, \quad h_\ell/\ell << 1, \quad \frac{\partial u}{\partial x} \approx 0$$

$$-\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2} = 0$$

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + Ay + B$$

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}(hy - y^2) + \frac{Uy}{h}$$



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Concluding Remarks

- Cartesian Coordinates
- Cylindrical Axial Flows
- Cylindrical Rotating Flows

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Thank you!

Questions?

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