ME 326 - Intermediate Fluid Mechanics — Clarkson University

Stream Function, Vorticity and Irrotational Flows

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Stream Function and Vorticity

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Outline

- Stream Function
- Vorticity
- Nondimensional Equations
- Creeping FLows
- Velocity Potential
- Irrotational Flows
- Flow Net

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Stream Function

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Incompressible Fluid

Cartesian Coordinates →

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Define Stream Function →

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

→

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} \equiv 0$$

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Streamlines

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$\psi = \text{Const.}$ → **Streamlines**

Streamlines are curves that are tangent to the velocity vector field.

To Find Streamline →

$$\frac{dx}{u} = \frac{dy}{v}$$

→

$$\frac{dx}{\frac{\partial \psi}{\partial y}} = \frac{dy}{-\frac{\partial \psi}{\partial x}}$$

→

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

→

$$\psi = \text{Const.}$$

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Stream Function

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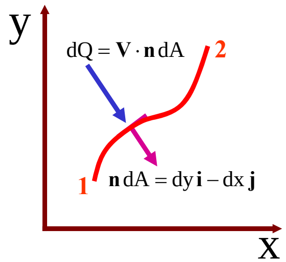
Flow Rate

$$dQ = \mathbf{V}\cdot\mathbf{n}dA = \left(\frac{\partial\psi}{\partial y}\mathbf{i} - \frac{\partial\psi}{\partial x}\mathbf{j}\right)\cdot(dy\mathbf{i} - dx\mathbf{j})$$

$$= \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx = d\psi$$



$$Q_{1-2} = \int_1^2 dQ = \int_1^2 d\psi = \psi_2 - \psi_1$$

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Stream Function

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Polar Coordinates **Incompressible Fluid**

$$\nabla\cdot\vec{V} = \frac{1}{r}\left[\frac{\partial(rv_r)}{\partial r} + \frac{\partial v_\theta}{\partial\theta}\right] = 0$$

Define Stream Function

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad v_\theta = -\frac{\partial\psi}{\partial r}$$

$$\frac{1}{r}\left[\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{\partial^2\psi}{\partial r\partial\theta}\right] \equiv 0$$

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Vorticity and Angular Velocity

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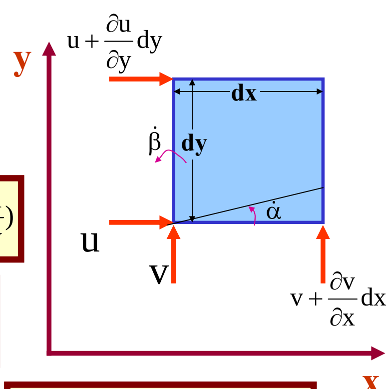
$$\dot{\alpha} = \frac{\partial v}{\partial x} \qquad \dot{\beta} = -\frac{\partial u}{\partial y}$$

Angular Velocity

$$\omega_z = \frac{1}{2}(\dot{\alpha} + \dot{\beta}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\boldsymbol{\omega} = \frac{1}{2}\nabla\times\mathbf{V} = \frac{1}{2}\mathrm{curl}\mathbf{V}$$



Vorticity

$$\zeta = 2\boldsymbol{\omega} = \nabla\times\mathbf{V} = \mathrm{curl}\mathbf{V}$$

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Navier-Stokes Equation

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Incompressible Fluids

$$\nabla\cdot\mathbf{V} = 0$$

$$\rho\left(\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = -\nabla P + \mu\nabla^2\mathbf{V} + \rho\mathbf{g}$$

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Dimensionless Equations

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Scaling

Length	L	{L}
Speed	U_∞	{L/T}
Frequency	f	{1/T}
Pressure	$P_o - P_\infty$	$\{M/(LT^2)\}$
Gravity	g	$\{L/T^2\}$

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Dimensionless Equations

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Dimensionless Variables

$$\mathbf{x}^* = \frac{\mathbf{x}}{L}$$

$$\mathbf{V}^* = \frac{\mathbf{V}}{U_\infty}$$

$$t^* = tf$$

$$P^* = \frac{P - P_\infty}{P_o - P_\infty}$$

$$\mathbf{g}^* = \frac{\mathbf{g}}{g}$$

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Dimensionless Equations

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Mass

$$\nabla^* \cdot \mathbf{V}^* = 0$$

Momentum

$$St \frac{\partial \mathbf{V}^*}{\partial t} + \mathbf{V}^* \cdot \nabla^* \mathbf{V}^* = -Eu \nabla^* P^* + \frac{1}{Re} \nabla^{*2} \mathbf{V}^* + \frac{1}{Fr^2} \mathbf{g}^*$$

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Dimensionless Groups

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Reynolds Number

$$\mathrm{Re} = \frac{\rho U_\infty L}{\mu}$$

Strouhal Number

$$St = \frac{fL}{U_\infty}$$

Euler Number

$$Eu = \frac{P_o - P_\infty}{\rho U_\infty^2}$$

Froude Number

$$Fr^2 = \frac{U_\infty^2}{gL}$$

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Creeping Flow Equations

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Mass

$$\nabla^* \cdot \mathbf{V}^* = 0 \qquad \nabla \cdot \mathbf{V} = 0$$

Momentum

For Re<<1, the Navier-Stokes Equation can be simplified to

$$-Eu\nabla^* P^* + \frac{1}{Re}\nabla^{*2}\mathbf{V}^* = 0 \qquad -\nabla P + \mu\nabla^2\mathbf{V} = 0$$

Drag on a sphere

$$F_D = 3\pi\mu V d$$

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Irrotational Flows

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A flow is irrotational if

$$\nabla \times \mathbf{V} = \mathrm{curl}\mathbf{V} = 0$$

For Two-Dimensional Flows

$$\zeta_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = -\nabla^2 \psi = 0$$

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Irrotational Flows

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For Irrotational Flows

$$\nabla^2 \psi = 0$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

Bernoulli's Eq.

Pressure

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{Const. (Every Where)}$$

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Irrotational Flows - Velocity Potential

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$$\nabla \times \mathbf{V} = 0 \;\Rightarrow\; \mathbf{V} = \nabla \phi$$

Velocity Potential

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

Conservation of Mass

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

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Irrotational Flows - Velocity Potential

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Polar Coordinates

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Incompressibility

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Irrotationality

$$\zeta_z = \frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

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Irrotational Flows

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Streamlines and Potential lines are related.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Streamlines ⟷ $\psi = \text{Const.}$

Potential lines ⟷ $\phi = \text{Const.}$

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Irrotational Flows

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Streamlines and Potential lines are orthogonal.

$$d\psi = 0 = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -v dx + u dy \;\Rightarrow\; \left.\frac{\partial y}{\partial x}\right|_\psi = \frac{v}{u}$$

$$d\phi = 0 = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = u dx - v dy \;\Rightarrow\; \left.\frac{\partial y}{\partial x}\right|_\phi = -\frac{u}{v}$$

$$\Rightarrow \left.\frac{\partial y}{\partial x}\right|_\phi = -\frac{1}{\left.\frac{\partial y}{\partial x}\right|_\psi}$$

⟷ **Orthogonal**

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Flow Net

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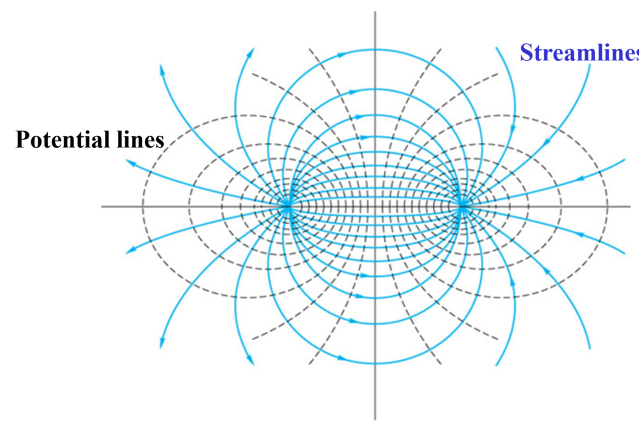

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Flow Net

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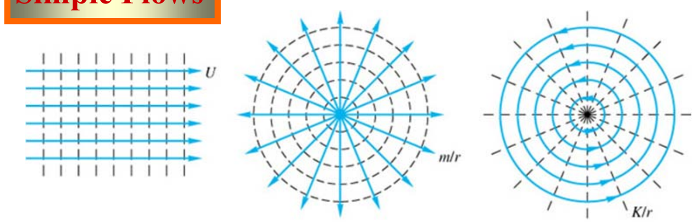
Simple Flows



Uniform Flow	Source Flow	Vortex Flow

Stream Function and Vorticity

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Concluding Remarks

- Stream Function & Streamlines
- Vorticity & Angular Velocity
- Velocity Potential
- Irrotational Flows
- Flow Net

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Thank you!

Questions?