Inviscid Irrotational Flows

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Outline

- Irrotational Flows
- Simple Flows
- Source/Sink Flows
- Vortex Flows
- Doublet Flows
- Flow Superposition
- Flow over a Cylinder

Euler Equation →

$$\rho\frac{d\mathbf{V}}{dt} = \underbrace{\rho\mathbf{g}}_{\text{Body Force}} - \underbrace{\nabla p}_{\text{Pressure Force}}$$

Continuity

$$\nabla\cdot\mathbf{V} = 0 \qquad \nabla^2\phi = 0$$

Given a ψ

Irrotationality

$$\nabla\times\mathbf{V} = 0 \qquad \nabla^2\psi = 0$$

Given a ϕ

Stream Function Formulation

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}$$

Potential Function Formulation

$$\mathbf{V} = \nabla\phi \qquad u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y}$$

Bernoulli's Eq.

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{Const.} \rightarrow P$$

Uniform Flows

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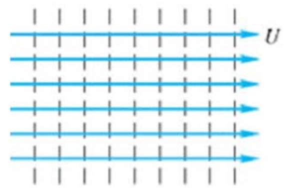
$$\phi = U_\infty x \qquad u = U_\infty$$

$$\psi = U_\infty y \qquad v = 0$$

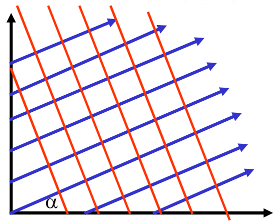


Uniform Flow at an Angle

$$u = U_\infty \cos\alpha \qquad v = U_\infty \sin\alpha$$

$$\phi = U_\infty (x\cos\alpha + y\sin\alpha)$$

$$\psi = U_\infty (y\cos\alpha - x\sin\alpha)$$



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Source/Sink Flows

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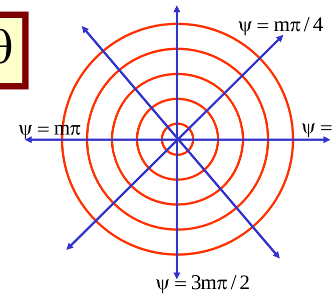
$$\phi = m\ln r \qquad \psi = m\theta$$

$$v_r = \frac{\partial\phi}{\partial r} = \frac{m}{r} \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0$$



Flow Rate

$$Q = \int_0^{2\pi} v_r r d\theta = 2\pi m$$

$$m = \frac{Q}{2\pi}$$

$$\phi = \frac{Q}{2\pi}\ln r \qquad \psi = \frac{Q}{2\pi}\theta$$

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Vortex Flows

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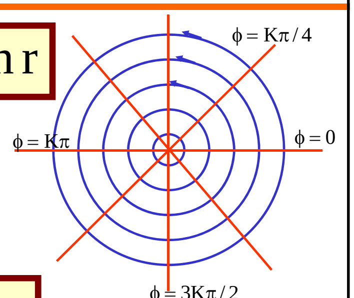
$$\phi = K\theta \qquad \psi = -K\ln r$$

$$v_r = \frac{\partial\phi}{\partial r} = 0 \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{K}{r}$$



Circulation

$$\Gamma = \oint_c \mathbf{V}\cdot\mathbf{ds} = \int_0^{2\pi} v_\theta r d\theta = 2\pi K$$

$$K = \frac{\Gamma}{2\pi} \qquad \phi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r$$

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Doublet Flows

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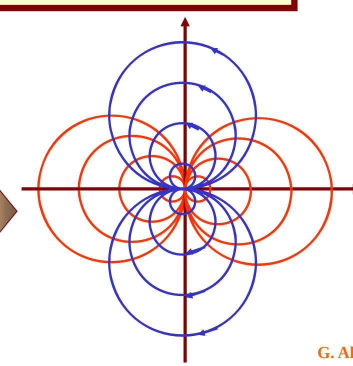
$$\phi = \frac{\lambda\cos\theta}{r} = \frac{\lambda x}{x^2+y^2} \qquad \psi = -\frac{\lambda\sin\theta}{r} = -\frac{\lambda y}{x^2+y^2}$$

Potential Lines

$$x^2 + y^2 - \frac{\lambda x}{\phi} = 0$$

Streamlines

$$x^2 + y^2 - \frac{\lambda y}{\psi} = 0$$



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Source, Vortex and Doublet at Arbitrary Points

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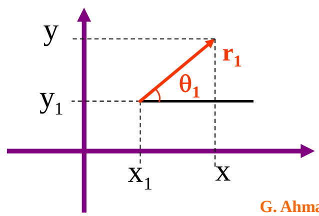
Source/Sink $\phi = m \ln r_1$ $\psi = m\theta_1$

Vortex $\phi = K\theta_1$ $\psi = -K \ln r_1$

Doublet $\phi = \frac{\lambda \cos\theta_1}{r_1} = \frac{\lambda (x - x_1)}{(x-x_1)^2 + (y-y_1)^2}$ $\psi = -\frac{\lambda \sin\theta_1}{r_1} = -\frac{\lambda (y - y_1)}{(x-x_1)^2 + (y-y_1)^2}$

$$r_1^2 = (x - x_1)^2 + (y - y_1)^2$$

$$\theta_1 = \tan^{-1}\frac{y - y_1}{x - x_1}$$



Superposition

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If ϕ_1 and ϕ_2 are velocity potentials, then

$$\nabla^2 \phi_1 = 0$$

$$\nabla^2 \phi_2 = 0$$

Let $\phi = \phi_1 + \phi_2$

$$\nabla^2 \phi = \nabla^2 (\phi_1 + \phi_2) = 0$$

$\phi = \phi_1 + \phi_2$ **Velocity Potential**

Source +Sink

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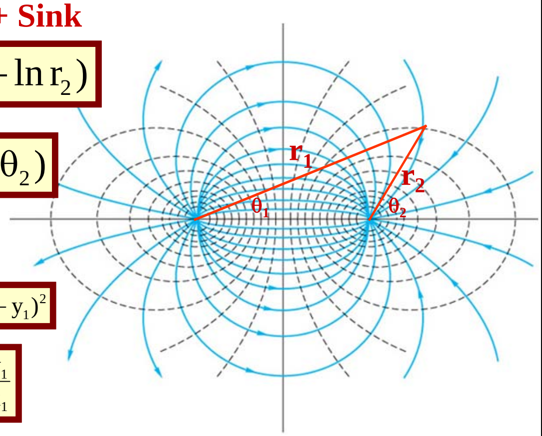
Source + Sink

$$\phi = m(\ln r_1 - \ln r_2)$$

$$\psi = m(\theta_1 - \theta_2)$$

$$r_1^2 = (x - x_1)^2 + (y - y_1)^2$$

$$\theta_1 = \tan^{-1}\frac{y - y_1}{x - x_1}$$



Rankin Half-Body

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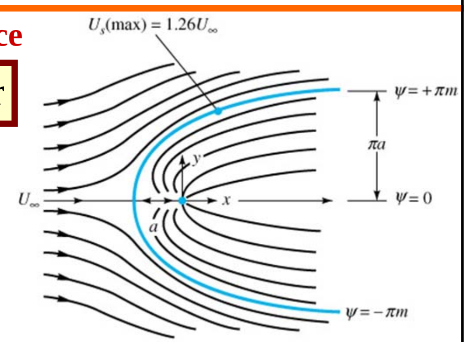
Uniform +Source

$$\phi = U_\infty x + m \ln r$$

$$\psi = U_\infty y + m\theta$$

$$u = \frac{\partial \phi}{\partial x} = U_\infty + \frac{mx}{r^2}$$

$$v = \frac{\partial \phi}{\partial y} = \frac{my}{r^2}$$



Stagnation Point $x = -\frac{m}{U_\infty} = -a$, $y = 0$

Stagnation Streamline $y = a(\pi - \theta)$

Flows Around a Cylinder

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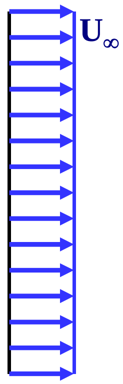
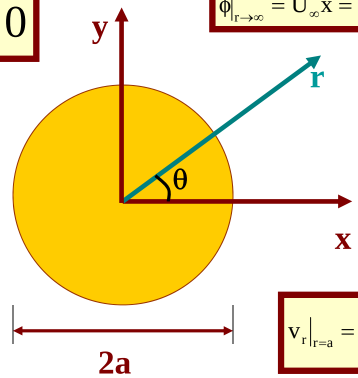


$$\nabla^2\phi = 0$$

$$\phi\big|_{r\to\infty} = U_\infty x = U_\infty r\cos\theta$$



$$v_r\big|_{r=a} = \frac{\partial\phi}{\partial r}\bigg|_{r=a} = 0$$

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Flows Around a Cylinder

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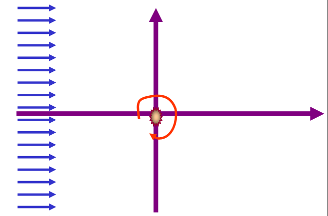
Uniform + Doublet+ Vortex

$$\phi = U_\infty r\cos\theta + \frac{\lambda\cos\theta}{r} + K\theta$$
$$= U_\infty\left(r + \frac{a^2}{r}\right)\cos\theta + K\theta$$



$$\psi = U_\infty r\sin\theta - \frac{\lambda\sin\theta}{r} - K\ln r$$
$$= U_\infty\left(r - \frac{a^2}{r}\right)\sin\theta - K\ln r$$

$$a^2 = \frac{\lambda}{U_\infty}$$

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Flows Around a Cylinder

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Velocity Field

$$v_r = \frac{\partial\phi}{\partial r} = U_\infty\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U_\infty\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{K}{r}$$

Stagnation Points

$$r = a$$
$$\theta = \sin^{-1}\frac{K}{2U_\infty a}$$

Stagnation Stream Function

$$\psi_{Stag} = -K\ln a$$

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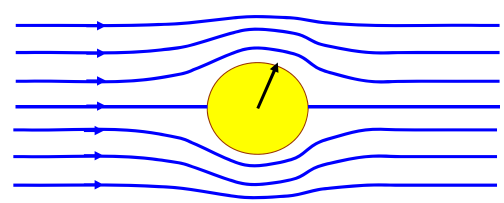
Stagnation Streamline

$$U_\infty\left(r - \frac{a^2}{r}\right)\sin\theta - K\ln\left(\frac{r}{a}\right) = 0$$

K=0



$$U_\infty\left(r - \frac{a^2}{r}\right)\sin\theta = 0$$

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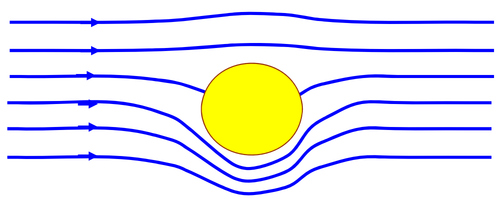
Stagnation Streamline

$$U_\infty\left(r-\frac{a^2}{r}\right)\sin\theta - K\ln\left(\frac{r}{a}\right)=0$$

K/2Ua<1



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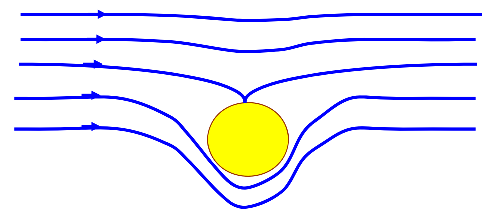
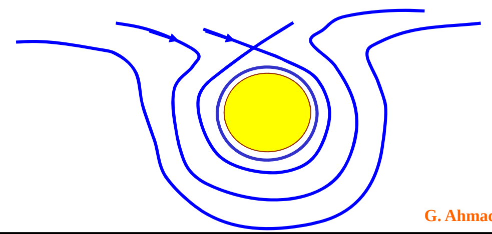
K/2Ua=1



K/2Ua>1



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Kutta-Joukowaski Lift

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At r=a

$$v_\theta = -2U_\infty\sin\theta+\frac{K}{a}$$

$$v_r=0$$

Bernoulli's Equation

$$\beta=\frac{K}{U_\infty a}$$

$$P=P_\infty+\frac{1}{2}\rho U_\infty^2(1-4\sin^2\theta+4\beta\sin\theta-\beta^2)$$

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Drag and Lift

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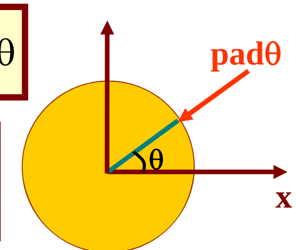
Drag

$$D=-\int_0^{2\pi}p\cos\theta a d\theta$$

$$D=0$$

Lift

$$L=-\int_0^{2\pi}p\sin\theta a d\theta$$



$$L=-4\beta a\frac{\rho U_\infty^2}{2}\int_0^{2\pi}\sin^2\theta d\theta$$

$$L=-\rho U_\infty 2\pi K=-\rho U_\infty\Gamma$$

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Drag and Lift

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Lift Coefficient

$$C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 (2a)} = \frac{\Gamma / a}{U_\infty} = \frac{2\pi K / a}{U_\infty} = \frac{2\pi v_\theta}{U_\infty}$$

Kutta-Joukowaski

$$L = \rho U_\infty \Gamma$$

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Method of Images

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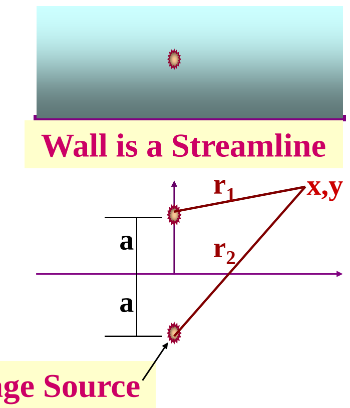
Source Near a Wall

Remove the wall but add an image source

$$\phi = m(\ln r_1 - \ln r_2)$$

$$r_1^2 = x^2 + (y-a)^2$$

$$r_2^2 = x^2 + (y+a)^2$$



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Superposition

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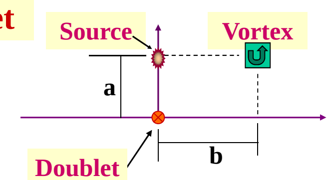
Source + Vortex+ Doublet

$$\phi = m\ln r_1 + K\theta_2 + \frac{\lambda\cos\theta}{r}$$



$$\phi = \frac{m}{2}\ln[x^2 + (y-a)^2] + K\tan^{-1}\frac{y-a}{x-b} + \frac{\lambda x}{x^2 + y^2}$$

$$u = \frac{\partial \phi}{\partial x} = \frac{m}{2}\frac{2x}{x^2 + (y-a)^2} + K\frac{-\frac{y-a}{(x-b)^2}}{1 + \left(\frac{y-a}{x-b}\right)^2} + \frac{\lambda(y^2 - x^2)}{(x^2 + y^2)^2}$$

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Concluding Remarks

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