ME 326 - Intermediate Fluid Mechanics

Clarkson University

Conservation of Mass and Balance of Momentum

Goodarz Ahmadi

Department of Mechanical and Aeronautical Engineering
Clarkson University
Potsdam, NY 13699-5725

Balance Laws

Clarkson University

Outline

- Conservation of Mass
- Balance of Momentum
- Navier-Stokes Equation

Differential Analysis

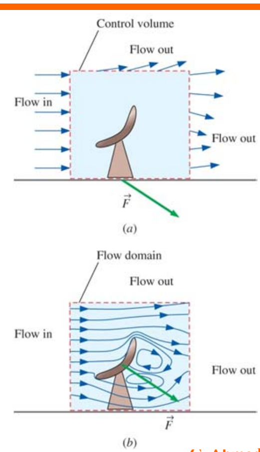
Clarkson University

Control Volume Analysis

- Inside is a black box

Differential Analysis

- Details at every point is evaluated



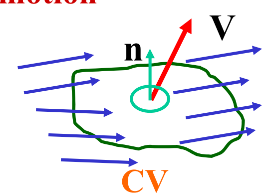
CONSERVATION of Mass

Clarkson University

Mass is conserved during the motion

Global

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_S \rho \mathbf{V} \cdot \mathbf{dS} = 0$$



Divergence Theorem

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_V \nabla \cdot (\rho \mathbf{V}) dV = 0$$

Local

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Conservation of Mass

Clarkson University

Compressible Fluid **Cartesian Coordinates**

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

Steady Flows

$$\nabla\cdot(\rho\mathbf{V})=\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

Conservation of Mass

Clarkson University

Incompressible Fluid

Cartesian Coordinates

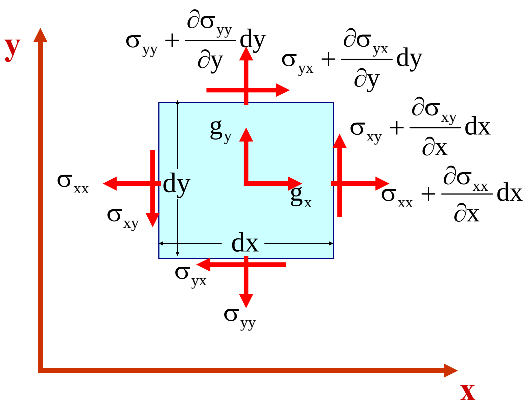
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla\cdot\vec{V}=0$$

Cylindrical (Polar) Coordinates

$$\nabla\cdot\vec{V}=\frac{1}{r}\frac{\partial(r v_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}=0$$

Balance of Momentum

Clarkson University



Balance of Momentum

Clarkson University

Newton's Law → $\sum\mathbf{F}=m\mathbf{a}=m\frac{d\mathbf{V}}{dt}$

x-direction → $\rho a_x=\rho g_x+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}$

y-direction → $\rho a_y=\rho g_y+\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}$

Balance of Momentum

Clarkson University

Cauchy's Law → $$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} + \nabla \cdot \boldsymbol{\sigma}$$

Constitutive Equations

Newtonian Fluid

$$\sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\sigma_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

ME 326 G. Ahmadi

Balance of Momentum

Clarkson University

Constitutive Equations →

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$

Navier-Stokes Equation (Incomp.)

$$\rho \frac{d\mathbf{V}}{dt} = \underbrace{\rho \mathbf{g}}_{\text{Body Force}} - \underbrace{\nabla p}_{\text{Pressure Force}} + \underbrace{\nabla^2 \mathbf{V}}_{\text{Viscous Force}}$$

ME 326 G. Ahmadi

Navier-Stokes Equation

Clarkson University

Incompressible Flows

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

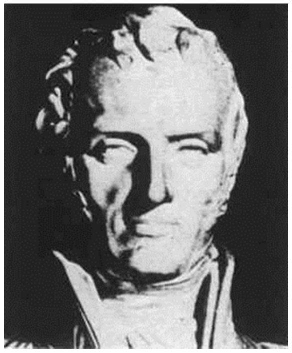
$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{V} = 0$$ ← **Mass**

ME 326 G. Ahmadi

Navier-Stokes Equation

Clarkson University



Navier

Stokes

ME 326 G. Ahmadi

Balance Laws

Clarkson University

Concluding Remarks

- Conservation of Mass
- Balance of Momentum
- Navier-Stokes Equation

ME 326 G. Ahmadi

Clarkson University

Thank you!

Questions?

ME 326 G. Ahmadi