Wiener-Hermite Expansions

Expansion of a function on an orthogonal set is one of the most common techniques of applied mathematics. Let $u(x)$ be an arbitrary function and the set $\{\varphi_n(x)\}$ be an orthogonal set. That is,

$$\int \varphi_n \varphi_m dx = \delta_{nm} \|\varphi_n\|, \tag{1}$$

$$\|\varphi_n\| = \int \varphi_n^2 dx. \tag{2}$$

Then

$$u(x) = \sum_n c_n \varphi_n(x), \qquad c_n = \frac{\int u\varphi_n dx}{\|\varphi_n\|}. \tag{3}$$

Expansion of a random function on a random base is developed by Wiener and Cameron and Martin. Let $a(x)$ be a white noise process with

$$< a(x) >= 0, \quad < a(x_1) a(x_2) >= \delta(x_1 - x_2). \tag{4}$$

A statistically orthogonal set may be constructed as

$$H^{(0)}(x) = 1, \qquad H^{(1)}(x) = a(x), \tag{5}$$

$$H^{(2)}(x_1, x_2) = a(x_1)a(x_2) - \delta(x_1 - x_2), \tag{6}$$

$$H^{(3)}(x_1, x_2, x_3) = a(x_1)a(x_2)a(x_3) - a(x_1)\delta(x_2 - x_3) - a(x_2)\delta(x_3 - x_1) - a(x_3)\delta(x_1 - x_2), \tag{7}$$

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and so on with

$$< H^{(i)} H^{(j)} >= 0 \qquad \text{for} \qquad i \neq j. \tag{8}$$

The set given by (5)-(8) is referred to as the Wiener- Hermite set. The is set is complete and

$$< H^{(0)}(x) H^{(0)}(x) >= 1, \tag{9}$$

$$< H^{(1)}(x_1)H^{(1)}(x_2) >= \delta(x_1 - x_2), \tag{10}$$

$$< H^{(2)}(x_1, x_2)H^{(2)}(x_3, x_3) >= \delta(x_1 - x_3)\delta(x_2 - x_4) + \delta(x_1 - x_4)(x_2 - x_3) \tag{11}$$

An arbitrary random function $u(x)$ may be expanded in terms of the Wiener-Hermite set. i.e.,

$$u(x) = \int K^{(1)}(x - x_1)H^{(1)}(x_1)dx_1 + \int\int K^{(2)}(x - x_1, x - x_2)H^{(2)}(x_1, x_2)H^{(2)}(x_1, x_2)dx_1dx_2 + \iiint K^3(x - x_1, x - x_2, x - x_3)H^{(3)}(x_1, x_2, x_3)dx_1dx_2dx_3 + \ldots \tag{12}$$

Here, the mean of $u(x)$ is taken to be zero,

$$< u(x) >= 0. \tag{13}$$

The first term of the series in Equation (12) is the Gaussian part of $u(x)$, the second and higher order terms are the non-Gaussian parts of $u(x)$.

Winer-Hermite Model for the Burger Equation

To illustrate the application of this method to turbulence, we consider the Burger model equation given as

$$\frac{\partial u(x,t)}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{14}$$

or

$$\left(\frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial x^2}\right)u + \frac{1}{2}\frac{\partial u^2}{\partial x^2} = 0. \tag{14}$$

Substituting the expansion given by (12) in Equation (14), we find

$$\left(\frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial x^2}\right)\left[\int K^{(1)}(x - x_1)H^{(1)}(x_1)dx_1 + \iint K^{(2)}(x - x_1, x - x_2)H^{(2)}(x_1, x_2)dx_1dx_2\right] + \frac{1}{2}\frac{\partial}{\partial x}[\iint K^{(1)}(x - x_1)\kappa^{(1)}(x - x_1')H^{(1)}(x_1)H^{(1)}(x_1')dx_1dx_1' \iiiint K^{(2)}(x - x_1, x - x_2)K^{(2)}(x - x_1', x - x_2')H^{(2)}(x_1, x_2)H^{(2)}(x_1', x_2')dx_1dx_2dx_1'dx_2' + 2\iiint K^{(1)}(x - x_1')K^{(2)}(x - x_1, x - x_2)H^{(1)}(x_1')H^{(2)}(x_1, x_2)dx_1dx_2dx_1'] = 0 \tag{15}$$

Multiplying (15) by $H^{(1)}(x')$ and $H^{(2)}(x', x'')$, respectively, and taking the expected value we find

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2}\right) K^{(1)}(x - x') + \frac{\partial}{\partial x}\Big[\int_{x_1} dx_1 K^{(1)}(x - x_1) K^{(2)}(x - x_1, x - x') + \int_{x_1} dx_1 K^{(1)}(x - x_1) K^{(2)}(x - x', x - x_1)\Big] = 0 \tag{16}$$

$$2\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2}\right) K^{(2)}(x - x', x - x'') + \frac{\partial}{\partial x}\left[K^{(1)}(x - x') K^{(1)}(x - x'')\right] = 0 \tag{17}$$

where $\left(K^{(2)}\right)^3$ terms were neglected. Rearranging (16) and (17) we find

$$\left\{ \begin{aligned} &\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2}\right) K^{(1)}(x - x') + 2\frac{\partial}{\partial x}\int dx_1 K^{(1)}(x - x_1) K^{(2)}(x - x_1, x - x') = 0 \\ &\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2}\right) K^{(2)}(x - x', x - x'') + \frac{1}{2}\frac{\partial}{\partial x}\int \left[K^{(1)}(x - x') K^{(1)}(x - x'')\right] = 0 \end{aligned} \right\} \tag{18}$$

These are two equations for finding two deterministic kernel functions $K^{(1)}$ and $K^{(2)}$.