ME 639 - Turbulence

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Karhunen-Loeve Orthagonal Expansion

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Karhunen-Loeve Expansion

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Outline

- Expansion of a function
- Orthonormal set
- Expansion of a random function
- K-L Expansion for periodic and non-periodic functions
- Response of linear system
- K-L expansion for Brownian motion

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Let $\varphi_n(t)$ be an orthonormal set

$$X(t) = \sum c_n \varphi_n(t)$$

$$c_n = \int_0^T X(t)\varphi_n(t)dt$$

$$\int_0^T \varphi_n(t)\varphi_m^*(t)dt = \delta_{nm}$$

Theorem

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In the expansion, the coefficients c_n become uncorrelated (orthogonal) random variables if and only if $\varphi_n(t)$ are the eigen-functions of the following Fredholm's integral equation:

$$\int_0^T R_{xx}(t_1, t_2)\varphi_n(t_2) = \lambda_n \varphi_n(t_1)$$

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$$E\{|c_n|^2\} = \lambda_n$$

$$E\{c_n c_m^*\} = E\{|c_n|^2\}\delta_{nm}$$

K-L Expansion converges in mean-square sense:

$$E\left\{\left[X(t) - \sum_n c_n \varphi_n(t)\right]^2\right\} = 0$$

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Autocorrelation

$$R_{xx}(t_1, t_2) = \sum_n \lambda_n \varphi_n(t_1)\varphi_n^*(t_2)$$

$$R_{xx}(t, t) = \sum_n \lambda_n |\varphi_n|^2$$

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Stationary and Periodic Processes

Stationary →

$$R_{xx} = R_{xx}(t_1 - t_2)$$

Periodic →

$$\varphi_n(t) = \frac{1}{\sqrt{T}} e^{in\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$x(t) = \sum_{-\infty}^{+\infty} \frac{c_n}{\sqrt{T}} e^{in\omega_0 t}$$

$$E\{|c_n|^2\} = \lambda_n$$

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Correlation and Spectrum

Correlation →

$$R_{xx}(t_1, t_2) = \frac{1}{T}\sum_{-\infty}^{+\infty} \lambda_n e^{in\omega_0 (t_1 - t_2)}$$

Spectrum →

$$S_{xx}(\omega) = \frac{1}{T}\sum_{-\infty}^{+\infty} \lambda_n \delta(\omega - n\omega_0)$$

$$E\{X^2(t)\} = \frac{1}{T}\sum_{-\infty}^{+\infty} \lambda_n$$

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Stationary Non-Periodic Processes

Expansion → $X(t) = \int_{-\infty}^{+\infty} e^{i\omega t} n(\omega)\sqrt{S(\omega)}\,d\omega$

n(ω) = White Noise → $E\{n(\omega_1)n(\omega_2)\} = \delta(\omega_1 - \omega_2)$

Correlation → $R_{xx}(t_1 - t_2) = \int_{-\infty}^{+\infty} e^{-i\omega(t_2 - t_1)} s(\omega)\,d\omega$

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Response of a Linear System

Linear System → $L_t X(t) = n(t)$

n(ω) = White Noise → $R_{nn}(t_1, t_2) = 2\pi S_0 \delta(t_1 - t_2)$

Response → $X(t) = \int_0^t h(t-\tau) n(\tau)\,d\tau$

Impulse Response → $L_t h(t) = \delta(t)$

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$$L_t R_{xx}(t, t_2) = L_t \int_0^t h(t-\tau) E\{n(\tau) X(t_2)\}\,d\tau$$

$$L_t R_{xx}(t, t_2) = E\{n(t) X(t_2)\} = 2\pi S_0 h(t_2 - t)$$

$$\int_0^T R_{xx}(t, t_2)\varphi(t_2)\,dt_2 = \lambda \varphi(t)$$

$$\int_0^T 2\pi S_0 h(t_2 - t)\varphi(t_2)\,dt_2 = \lambda L_t \phi(t)$$

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$$\lambda L_{-t} L_t \phi(t) = \int_0^T 2\pi S_0 \delta(t_2 - t)\phi(t_2)\,dt_2 = 2\pi S_0 \varphi(t)$$

$$\varphi^{(i)}(0) = 0$$

$$L_t \varphi^{(i)}(t)|_{t=T} = 0$$

$$i = 0, 1, \ldots, N-1$$

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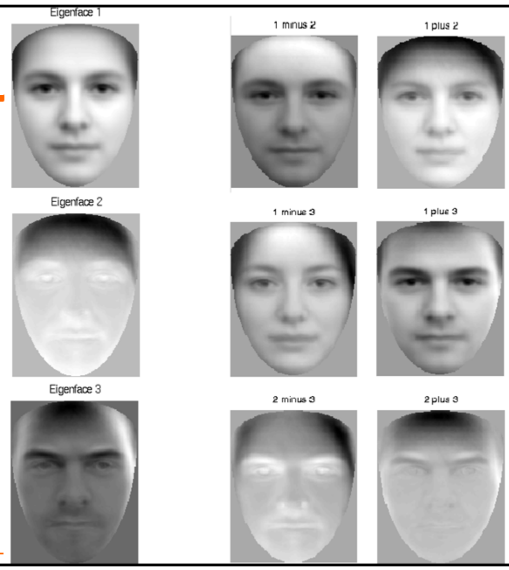

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Concluding Remarks

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Thank you!

Questions?