INTRODUCTION TO TURBULENCE MODELING

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In this section, an introduction to the historical development in turbulence modeling is provided.

Outline

- Viscous Fluid
- Turbulence
- Classical Phenomenological models (Mixing Length)
- One-Equation Models
- Two-Equation Models (The $k-e$ Model)
- Stress Transport Models
- Material-Frame Indifference
- Continuum Approach
- Consistency of the $k-e$ Model
- Rate-Dependent Model

VISCOUS FLOW

The conservation laws for a continuous media are:

Mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum

$$\rho \frac{d\mathbf{u}}{dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{t}$$

Angular Momentum

$$\mathbf{t}^{T} = \mathbf{t}$$

Energy

$$\rho \dot{e} = \mathbf{t} : \nabla \mathbf{u} + \nabla \mathbf{q} + \rho h$$

Entropy Inequality

$$\rho \dot{\eta} - \nabla \cdot \left(\frac{\mathbf{q}}{T}\right) - \frac{\rho h}{T} \geq 0$$

Constitutive Equation

Experimental evidence shows that for a viscous fluid, the stress is a function of velocity gradient. That is

$$\tau_{kl} = -p\delta_{kl} + G_{kl}(u_{i,j})$$

The velocity gradient term may be decomposed as

$$u_{i,j} = d_{ij} + \omega_{ij}$$

where d_{ij} is the deformation rate tensor and w_{ij} is the spin tensor. These are given as

$$d_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \qquad \omega_{kl} = \frac{1}{2}(u_{k,l} - u_{l,k})$$

The principle of Material Frame-Indifference of continuum mechanics implies that the stress is generated only by the deformation rate of media and the spin has no effect. This is because both stress and deformation rate tensors are frame-indifferent while spin is not. Thus, the general form of the constitutive equation is given as

$$\tau_{kl} = -p\delta_{kl} + F_{kl}(d_{ij})$$

For a Newtonian fluid, the constitutive equation is linear and is given as

$$\tau_{kl} = (-p + \lambda u_{i,i})\delta_{kl} + 2\mu d_{kl}$$

The entropy inequality imposed the following restrictions on the coefficient of viscosity:

$$3\lambda + 2\mu \geq 0, \ \mu \geq 0,$$

Using the constitutive equation in the balance of momentum leads to the celebrated Navier-Stokes equation. For an incompressible fluid the Navier-Stokes and the continuity equations are given as

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j},$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

These form four equations for evaluating four unknowns u_i, p.

TURBULENT FLOW

In turbulent flows the field properties become random functions of space and time. Thus

$$u_i = U_i + u_i' \qquad \overline{u}_i = U_i, \ \overline{u_i'} = 0$$

$$p = P + p' \qquad \overline{p} = P, \ \overline{p'} = 0$$

Substituting the decomposition into the Navier-Stokes equation and averaging leads to the Reynolds equation.

Reynolds Equation

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

Here

$\tau_{ij}^T = -\rho \overline{u_i' u_j'}$ =Turbulent Stress Tensor

First Order Modeling (Classical Phenomenology)

<u>Boussinesq Eddy Viscosity:</u>

$$\tau_{21}^T = -\rho \overline{u'v'} = \rho \nu_T \frac{dU}{dy}$$

$$\frac{\tau_{ij}^T}{\rho} = -\overline{u_i' u_j'} = \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{1}{3}\overline{u_k' u_k'}\,\delta_{ij}$$

<u>Prandtl Mixing Length</u>

$$\tau_{21}^T = \rho l_m^2 \left|\frac{\partial U}{\partial y}\right| \frac{\partial U}{\partial y}$$

$$\nu_T = l_m^2 \left|\frac{\partial U}{\partial y}\right|, \qquad -\overline{T'v'} = \frac{\nu_T}{\sigma_T}\frac{\partial T}{\partial y}$$

<u>Kolmogorov-Prandtl Expression</u>

Eddy Viscosity

$\nu_T \approx c u \ell$, u = velocity scale, ℓ = length scale, c = const.

Kinematic Viscosity

$\nu \propto c\lambda$, c = speed of sound, λ = mean free path

Let

$$u \sim \ell_m \left|\frac{\partial U}{\partial y}\right|, \qquad \ell = \ell_m \Rightarrow \nu_T = \ell_m^2 \left|\frac{\partial U}{\partial y}\right|$$

For free shear flows

$\ell_m \simeq c\ell_0$ (ℓ_0 = half width)

Close to a wall

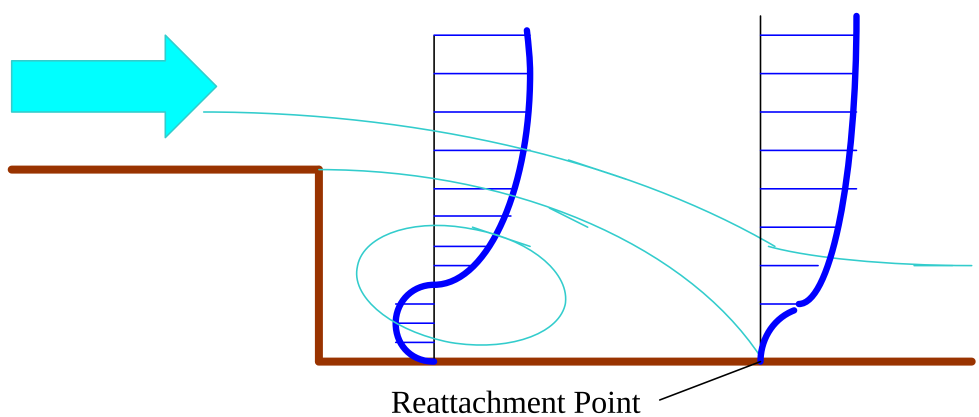
$\ell_m = \kappa y$ (y = distance from the wall)

Local Equilibrium

For local equilibrium Production = Dissipation	$\Leftrightarrow$	Mixing length Hypothesis

Shortcomings of the Mixing Length Model

- When $\frac{\partial U}{\partial y} = 0 \Rightarrow \nu_T = 0$
- Lack of transport of scales of turbulence
- Estimating the mixing length, ℓ_m.



Schematics of flow over a backward facing step.

At the reattachment point $\frac{\partial U}{\partial y} = 0$, which leads to vanishing eddy diffusivity and thus negligible heat flux. Experiments, however, show that the heat flux becomes maximum at the reattachment point.

One-Equation Models

Eddy Viscosity

$$\nu_T = c_\mu k^{1/2} \ell, \qquad k = \frac{1}{2}\overline{u_i' u_i'} = \text{Turbulence Kinetic Energy}$$

Exact k-equation

$$\underbrace{\frac{d}{dt}\frac{\overline{u_i' u_i'}}{2}}_{\text{Convective Transport}} = \underbrace{-\frac{\partial}{\partial x_k}\overline{u_k'\left(\frac{u_i' u_i'}{2} + \frac{P'}{\rho}\right)}}_{\text{Turbulence Diffusion}} \underbrace{-\overline{u_i' u_j'}\frac{\partial U_i}{\partial x_j}}_{\text{Production}} \underbrace{-\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}}_{\text{Dissipation}} + \underbrace{\nu\frac{\partial^2}{\partial x_j x_j}\frac{\overline{u_i' u_i'}}{2}}_{\text{Viscous Diffusion}}$$

where

$$\frac{d}{dt}\frac{\overline{u_i' u_i'}}{2} = \text{convective transport}, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + U_j\frac{\partial}{\partial x_j}$$

$$\frac{\partial}{\partial x_k}\overline{u_k'\left(\frac{u_i' u_i'}{2} + \frac{P'}{\rho}\right)} = \text{turbulence diffusion}$$

$$\overline{u_i' u_j'}\frac{\partial U_i}{\partial x_j} = \text{production}$$

$$\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} = \ , = \text{dissipation}$$

$$\nu\frac{\partial^2}{\partial x_j x_j}\frac{\overline{u_i' u_i'}}{2} = \text{viscous diffusion}$$

Modeled k-equation

$$\frac{dk}{dt} = \frac{\partial}{\partial x_j}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_j}\right) + \nu_T\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - c_D\frac{k^{3/2}}{\ell},$$

where

$$\frac{\partial}{\partial x_j}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_j}\right) = \text{turbulence diffusion}, \qquad \sigma_k \approx 1 \text{ (turbulence Prandlt number)}$$

$$\nu_T\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} = \text{production},$$

$$c_D \frac{k^{3/2}}{\ell} = \varepsilon = \text{dissipation.}$$

Note that the turbulence length scale ℓ is given by an algebraic equation.

Bradshaw's Model

Modeled k-equation

$$\frac{dk}{dt} = \frac{\partial}{\partial y}(Bk\sqrt{\frac{\tau_{Max}}{\rho}}) + ak\frac{\partial U}{\partial y} - c_D \frac{k^{3/2}}{\ell}$$

where

$$ak\frac{\partial U}{\partial y} = \text{production}$$

$$c_D \frac{k^{3/2}}{l} = \text{dissipation}$$

$$-\overline{u'v'} = ak \quad \text{(shear stress} \propto \text{kinetic energy)}$$

$$B = \frac{\tau_{Max}}{\rho v_0^2} g(\frac{y}{\delta}), \; \ell = \delta f(\frac{y}{\delta})$$

Shortcomings of One-Equation Models

- Use of an algebraic equation for the length scale is too restrictive.
- Transport of the length scale is not accounted for.

Transport of Second Scale (Boundary Layer)

The transport of a second turbulence scale, z, is given as

z-Equation

$$\frac{dz}{dt} = \frac{\partial}{\partial y}(\frac{\nu_T}{\sigma_z}\frac{\partial z}{\partial y}) + z[c_1 \frac{\nu_T}{k}(\frac{\partial U}{\partial y})^2 - c_2 \frac{k}{\nu_T}] + S_z$$

where

$$\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_z}\frac{\partial z}{\partial y}\right) = \text{diffusion}$$

$$c_1\frac{\nu_T}{k}\left(\frac{\partial U}{\partial y}\right)^2 = \text{production}$$

$$c_2\frac{k}{\nu_T} = \text{destruction,}$$

$$S_z = \text{secondary source}$$

Choices for z

$$\text{Turbulence Time Scale} = \sqrt{\ell^2/k}$$

$$\text{Turbulence frequency Scale} = \sqrt{k/\ell^2}$$

$$\text{Turbulence mean-square vorticity Scale} = k/\ell^2$$

$$\text{Turbulence Dissipation} = \quad \varepsilon = \nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_i}}$$

$$z = k\ell$$

e -Equation (exact):

$$\frac{d\varepsilon}{dt} = \underbrace{-\frac{\partial}{\partial x_j}(\overline{u_j'\varepsilon'})}_{\text{Diffusion}} \underbrace{- 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_l}\frac{\partial u_k'}{\partial x_l}}}_{\text{Generation by vortex stretching}} \underbrace{- 2\nu\overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_l}\frac{\partial^2 u_i'}{\partial x_k \partial x_l}}}_{\text{Viscous destruction}}$$

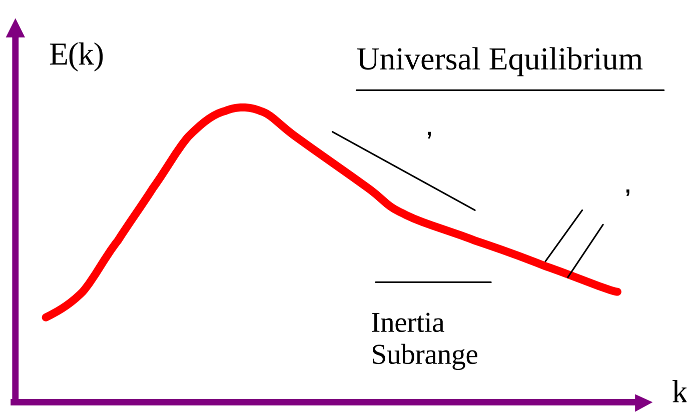
Note that

$$\varepsilon \sim \frac{k^{3/2}}{\ell}, \qquad \ell \sim \frac{k^{3/2}}{\varepsilon}$$

Thus

$$\nu_T \sim \sqrt{k}\,\ell \sim \frac{k^2}{\varepsilon}$$

Note also that ε is also the amount of energy that paths through the entire spectrum of eddies of turbulence.



Schematics of turbulence energy spectrum.

Two-Equation Models

The $k-\varepsilon$ Model

$$\nu_T = \frac{c_\mu k^2}{\varepsilon}, \qquad -\overline{u_i' u_j'} = \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3} k \delta_{ij}$$

k-equation

$$\frac{dk}{dt} = \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\upsilon_T}{\sigma_k}\frac{\partial k}{\partial x_j}\right)}_{\text{Diffusion}} + \underbrace{\nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}}_{\text{Production}} - \underbrace{\varepsilon}_{\text{dissipation}}$$

ε*-equation*

$$\frac{d\varepsilon}{dt} = \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\upsilon_T}{\sigma_\varepsilon}\frac{\partial k}{\partial x_j}\right)}_{\text{Diffusion}} + \underbrace{c_{\varepsilon 1} \nu_T \frac{\varepsilon}{k}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}}_{\text{Generation}} - \underbrace{c_{\varepsilon 2}\frac{\varepsilon^2}{k}}_{\text{Distruction}},$$

where

$c_\mu = 0.09$, $c_{\varepsilon 1} = 1.45$, $c_{\varepsilon 2} = 1.9$, $s_k = 1$, $\sigma_\varepsilon = 1.3$, (Jones and Launder, 1973)

Momentum

$$\frac{dU_i}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_j}\overline{u_i' u_j'}$$

Mass

$$\frac{\partial U_i}{\partial x_i} = 0$$

Closure: Six Equations for six unknowns, v_i, P, k, ε.

Kolmogorov Model

$$\frac{dk}{dt} = \underbrace{2\upsilon_T S_{ij} S_{ij}}_{\text{Production}} - \underbrace{\frac{1}{2}k^2\omega}_{\text{Dissipation}} + \underbrace{A'' \frac{\partial}{\partial x_j}\left(\frac{k}{\omega}\frac{\partial k}{\partial x_j}\right)}_{\text{Diffusion}}$$

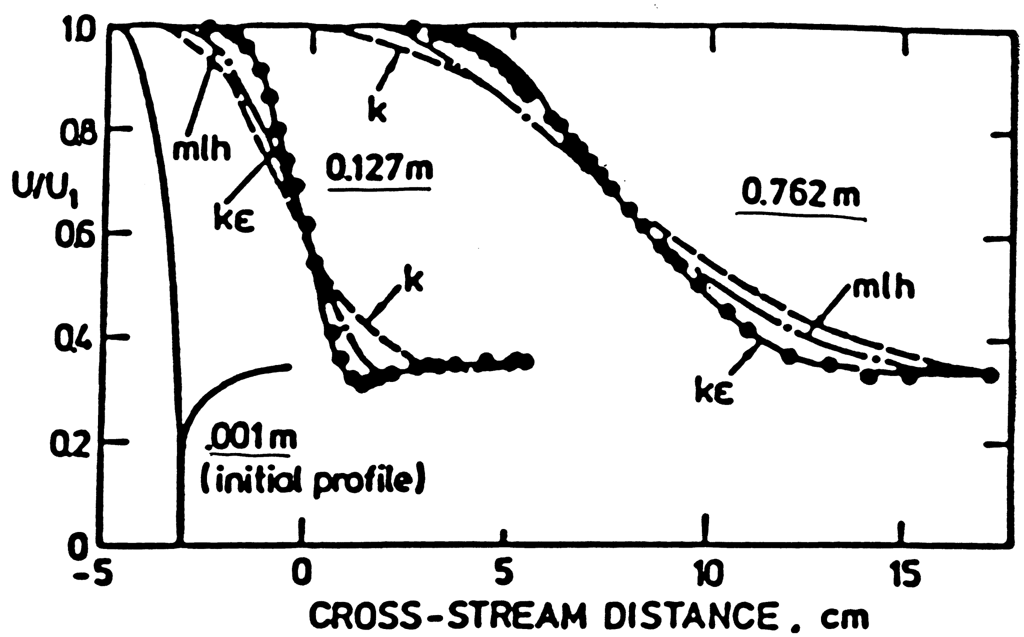
$$\frac{dw}{dt} = -\frac{7}{10}\omega^2 + 2A'\frac{\partial}{\partial x_j}\left(\frac{k}{\omega}\frac{\partial k}{\partial x_j}\right)$$

$$\nu_T = \frac{Ak}{\omega}, \; S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

Comparison of Model Predictions

In this section comparisons of the predictions of the mixing length and one and two-equation models with the experimental data for simple turbulent shear flows are presented.

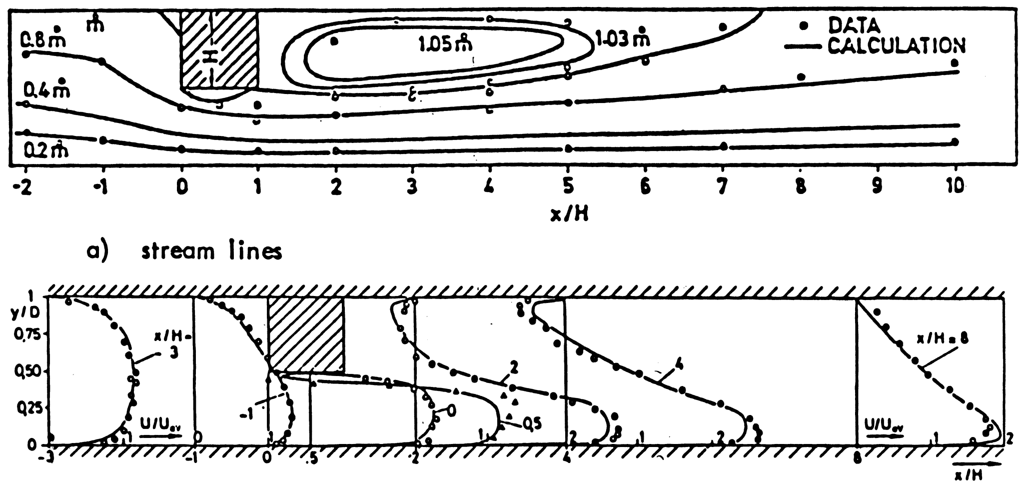
Development of Plane Mixing Layer (Rodi, 1982)



It is seen that the $k-\varepsilon$ model captures the features of the flow more accurately when compared with the one-equation and mixing length model.

Turbulent Recirculating Flow (Durst and Rastogi, 1979)

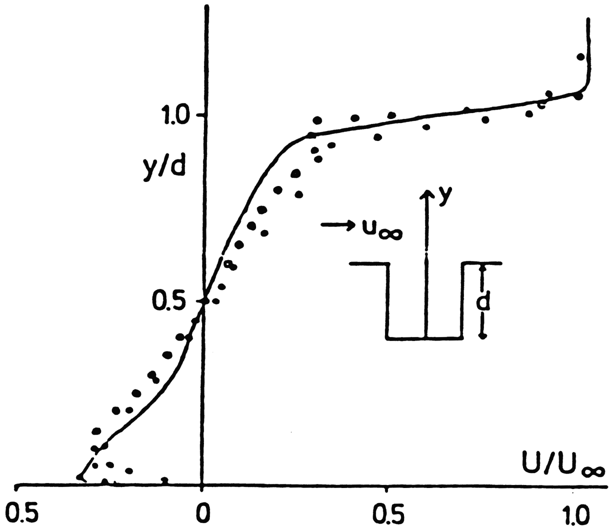
The $k-\varepsilon$ model predictions for turbulent flow in a channel with an obstructing block are compared with the experimental data.

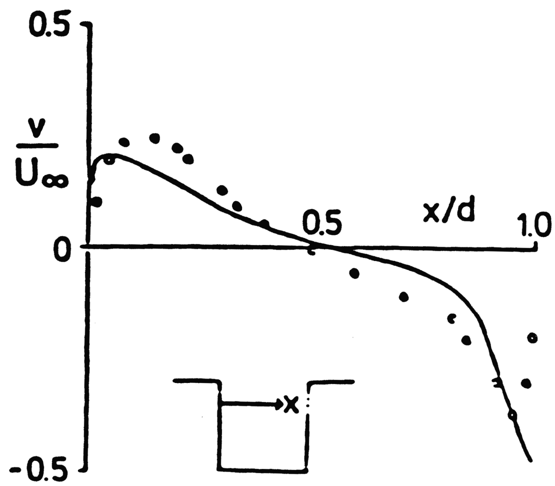


b) Velocity profiles

Flow in a Square Cavity (Gosman and Young)

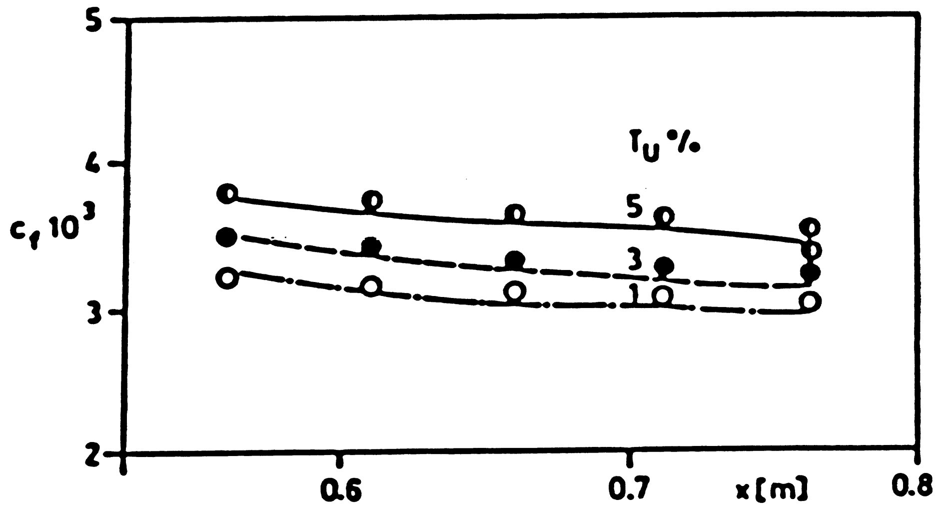
The $k-\varepsilon$ model predictions for a square cavity are shown in this section.





Free-Stream Turbulence

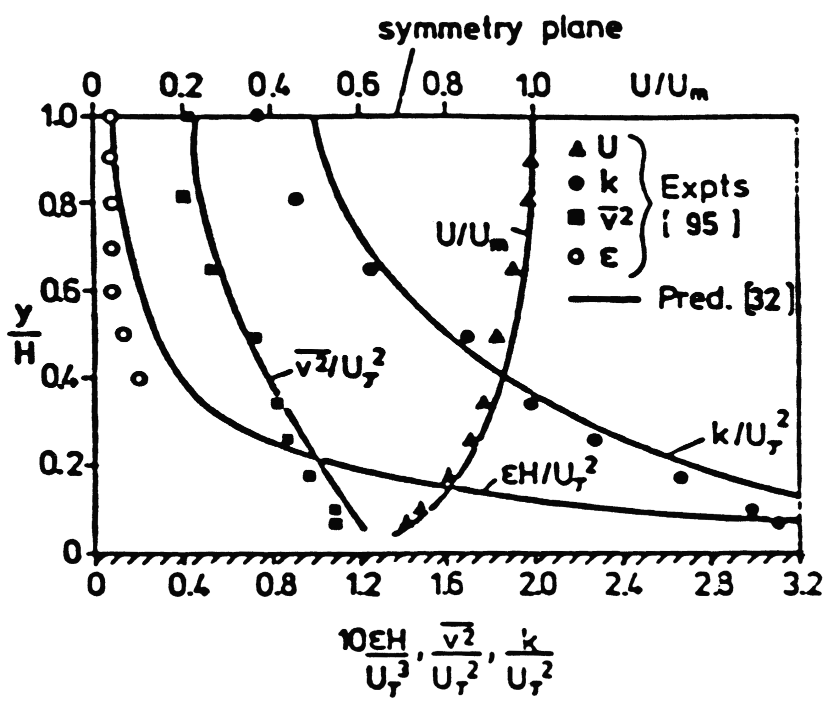
The free stream turbulence affects the skin friction coefficient. The mixing length model can not predict such effects. The $k-\varepsilon$ model does a reasonable job in predicting the increase of skin friction coefficient with the free stream turbulence.



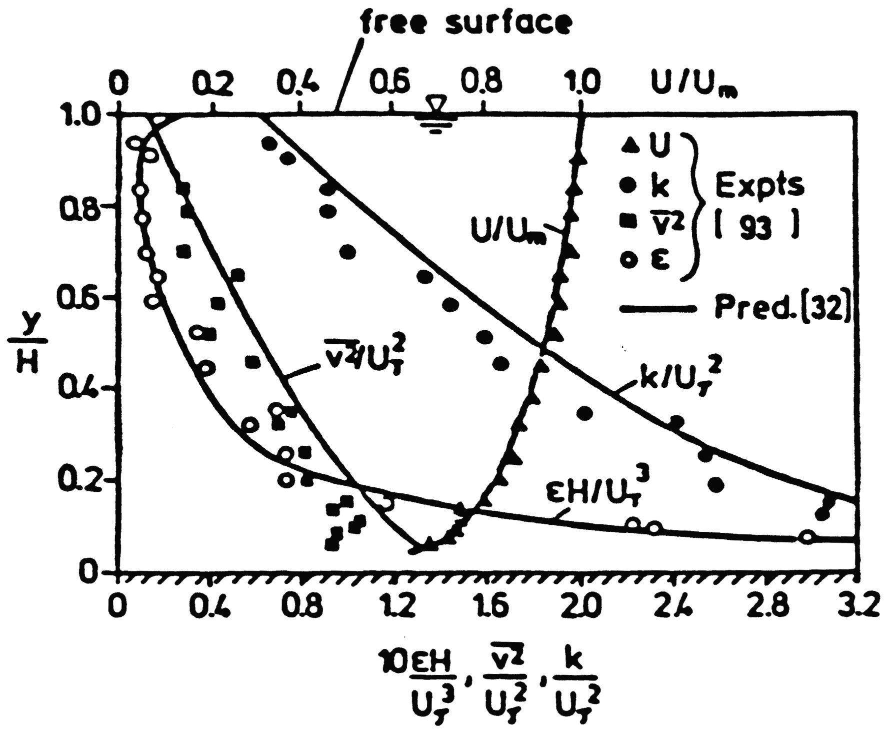
Turbulent Channel Flow (Rodi, 1980)

Distribution of mean velocity and turbulence quantities in fully developed two-dimensional channel flows was predicted by Rodi (1980) using an algebraic stress model (a modified $k-e$ model)

Closed Channel Flow



Open Channel Flow

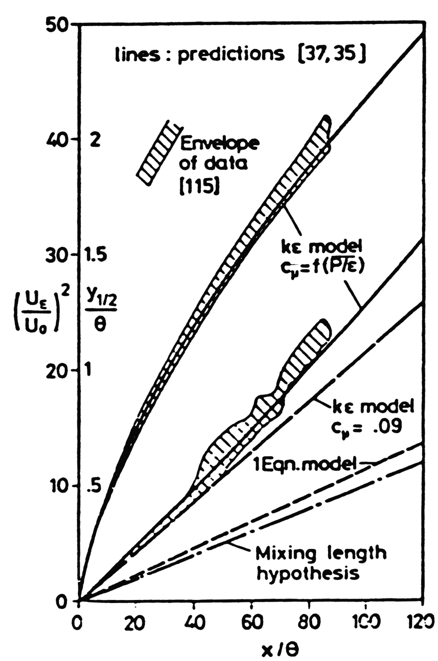


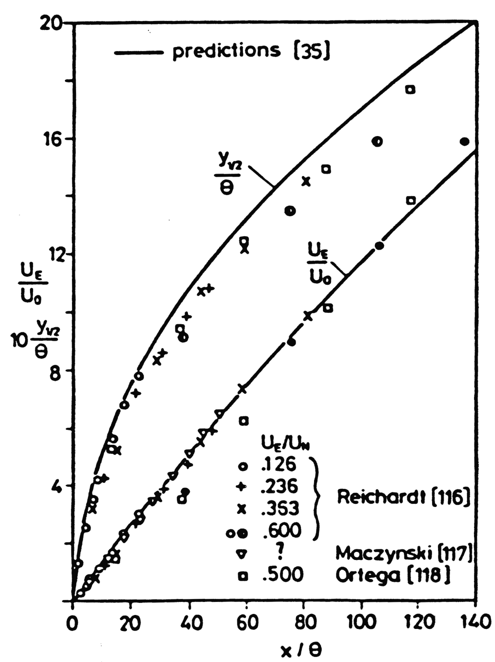
Jets Issuing in Co-flowing Streams (Rodi, 1982)

For a jet, $\theta = \frac{1}{U_E}\int_0^{\infty} U(U - U_E)dy = \text{Excess momentum thickness}$

Plane Jet

Round Jet





Shortcomings of the $k-e$ Model

$$-\overline{u_i' u_j'} = \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3} k \delta_{ij}$$

- Limited to an eddy viscosity assumption.
- Eddy viscosity and diffusivity are assumed to be isotropic.
- Convection and diffusion of the shear stresses are neglected.
- Normal turbulent stresses are not considered.
- Main assumption is: $\overline{u_i' u_j'} \sim k$.

Stress Transport Models

Subtracting the Navier-Stokes equation form the Reynolds equation, we find an evolution equation for the turbulence fluctuation velocity. That is

$$\frac{\partial u_i'}{\partial t} + U_k \frac{\partial u_i'}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k}\overline{u_i' u_k'} - \frac{\partial}{\partial x_k}(u_i' u_k') - u_k' \frac{\partial U_i}{\partial x_k} \quad (1)$$

$$\frac{\partial u_j'}{\partial t} + U_k \frac{\partial u_j'}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \nu \frac{\partial^2 u_j'}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k}\overline{u_j' u_k'} - \frac{\partial}{\partial x_k}(u_j' u_k') - u_k' \frac{\partial U_j}{\partial x_k} \quad (2)$$

Multiplying Equation (1) by u_j', and Equation (2) by u_i', adding the resulting equations and averaging leads to the exact stress transport equation:

$$\underbrace{\left(\frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k}\right)\overline{u_i' u_j'}}_{\text{Convection}} = \underbrace{-\left[\overline{u_i' u_k'}\frac{\partial U_j}{\partial x_k} + \overline{u_j' u_k'}\frac{\partial U_i}{\partial x_k}\right]}_{\text{Production}} \underbrace{- 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}}_{\text{Dissipation}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}}_{\text{Pressure-strain}}$$

$$\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{u_i' u_j' u_k'} + \overline{\frac{p'}{\rho}(u_i' \delta_{jk} + u_j' \delta_{ik})} - \nu \frac{\partial}{\partial x_k}\overline{u_i' u_j'}\right]}_{\text{Diffusion}},$$

where

$$\left[\overline{u_i' u_k'}\frac{\partial U_j}{\partial x_k} + \overline{u_j' u_k'}\frac{\partial U_i}{\partial x_k}\right] = \text{production},$$

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$$2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} = \text{dissipation},$$

$$\overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)} = \text{pressure strain}$$

$$\frac{\partial}{\partial x_k}\left[\overline{u_i'u_j'u_k'}+\overline{\frac{p'}{\rho}(u_i'\delta_{jk}+u_j'\delta_{ik})}-\nu\frac{\partial}{\partial x_k}\overline{u_i'u_j'}\right] = \text{diffusion.}$$

Modeling Diffusion:

$$-\overline{u_i'u_j'u_k'} = c_s\frac{k}{\varepsilon}\left(\overline{u_i'u_l'}\frac{\partial\overline{u_i'u_k'}}{\partial x_l}+\overline{u_j'u_l'}\frac{\partial\overline{u_k'u_i'}}{\partial x_l}+\overline{u_k'u_l'}\frac{\partial\overline{u_i'u_j'}}{\partial x_l}\right)$$

Pressure diffusion ≈ 0
Viscous diffusion ≈ 0

Modeling Dissipation

$$2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} = \frac{2}{3}\delta_{ij}\varepsilon$$

Modeling Pressure-Strain

$$\overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)} = -\int_{\mathbf{x}_1} d\mathbf{x}_1\, G(\mathbf{x},\mathbf{x}_1)\Big\{ \underbrace{\overline{\left(\frac{\partial u_l'}{\partial x_m}\frac{\partial u_m'}{\partial x_l}\right)_1\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)}}_{\varphi_{ij}^{(1)}}$$

$$+\underbrace{2\left(\frac{\partial U_l}{\partial x_m}\right)_1\overline{\left(\frac{\partial u_m'}{\partial x_l}\right)_1\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)}}_{\varphi_{ij}^{(2)}+\varphi_{ji}^{(2)}}\Big\}$$

where

$$\varphi_{ij}^{(1)} = -c_1\left(\frac{\varepsilon}{k}\right)\left(\overline{u_i'u_j'}-\frac{2}{3}\delta_{ij}k\right) \ \text{(Return to isotropy)}$$

$$\varphi_{ij}^{(2)}+\varphi_{ji}^{(2)} = -\gamma\left(P_{ij}-\frac{2}{3}P\delta_{ij}\right) \ \text{(Rapid term)}$$

Here

$$P_{ij} = -\left(\overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k}+\overline{u_i'u_k'}\frac{\partial U_i}{\partial x_k}\right)$$

Pressure-Strain Correlation

Modeling the pressure-strain correlation, $\overline{\frac{p'}{\rho}(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i})}$, is critical to the stress transport equations.

Navier-Stokes Equation

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j} \tag{1}$$

Taking the divergence of (1), we find

$$\frac{\partial^2 u_i u_k}{\partial x_i\partial x_k}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i\partial x_i}, \tag{2}$$

or

$$\nabla^2\frac{p}{\rho}=-\frac{\partial^2 u_i u_k}{\partial x_i\partial x_k}. \tag{3}$$

Averaging Equation (3), the result is:

$$\nabla^2\frac{\bar{p}}{\rho}=-\frac{\partial^2 U_i U_k}{\partial x_i\partial x_k}-\frac{\partial^2\overline{u_i'u_k'}}{\partial x_i\partial x_k}. \tag{4}$$

Subtracting (4) from (3), we find

$$\nabla^2\frac{p'}{\rho}=-2\frac{\partial U_i}{\partial x_k}\frac{\partial u_k'}{\partial x_i}-\frac{\partial u_i'}{\partial x_k}\frac{\partial u_k'}{\partial x_i}+\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_k'}{\partial x_i}}. \tag{5}$$

Introducing the Green function $G(\mathbf{x},\mathbf{x_1})$ for the Poisson equation. i.e.,

$$\nabla^2 G(\mathbf{x},\mathbf{x_1})=\delta(\mathbf{x}-\mathbf{x_1}), \tag{6}$$

Equation (5) may be restated as

$$p' = -\int_{\mathbf{x}_1} G(\mathbf{x}, \mathbf{x}_1)[2\frac{\partial U_i}{\partial x_k}\frac{\partial u'_k}{\partial x_i} + \frac{\partial u'_i}{\partial x_k}\frac{\partial u'_k}{\partial x_i} - \overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_k}{\partial x_i}}]d\mathbf{x}_1 \tag{7}$$

The pressure-strain rate correlation then becomes

$$\overline{\frac{p'}{\rho}(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i})} = -\int_{\mathbf{x}_1} G(\mathbf{x}, \mathbf{x}_1)[\overline{(\frac{\partial u'_l}{\partial x_k}\frac{\partial u'_k}{\partial x_l})_1(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i})} + 2(\frac{\partial U_l}{\partial x_k})_1(\frac{\partial u'_k}{\partial x_l})_1(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i})]d\mathbf{x}_1 \tag{8}$$

or

$$\overline{\frac{p'}{\rho}(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i})} = \varphi_{ij}^{(1)} + (\varphi_{ij}^{(2)} + \varphi_{ji}^{(2)}) \tag{9}$$

Note that for unbounded regions

$$G(\mathbf{x}, \mathbf{x}_1) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x} - \mathbf{x}_1|}. \tag{10}$$

Modeled Stress Transport Equation

$$\underbrace{(\frac{\partial}{\partial t} + U_k\frac{\partial}{\partial x_k})\overline{u'_i u'_j}}_{\text{Convection}} = \underbrace{-[\overline{u'_j u'_k}\frac{\partial U_i}{\partial x_k} + \overline{u'_i u'_k}\frac{\partial U_j}{\partial x_k}]}_{\text{Production}} - \underbrace{\frac{2}{3}}_{\text{Dissipation}}\delta_{ij}\varepsilon$$

$$\underbrace{-c_1\frac{\varepsilon}{k}(\overline{u'_i u'_j} - \frac{2}{3}\delta_{ij}k) + (\varphi_{ij}^{(2)} + \varphi_{ji}^{(2)})}_{\text{Pressure–strain}} + \underbrace{(\varphi_{ij}^{(w)} + \varphi_{ji}^{(w)})}_{\text{Wall effects}}$$

$$\underbrace{+c_s\frac{\partial}{\partial x_k}\{\frac{k}{\varepsilon}[\overline{u'_i u'_l}\frac{\partial\overline{u'_j u'_k}}{\partial x_l} + \overline{u'_j u'_l}\frac{\partial\overline{u'_k u'_i}}{\partial x_l} + \overline{u'_k u'_l}\frac{\partial\overline{u'_i u'_j}}{\partial x_l}]\}}_{\text{Diffusion}}$$

where

$$[\overline{u'_j u'_k}\frac{\partial U_i}{\partial x_k} + \overline{u'_i u'_k}\frac{\partial U_j}{\partial x_k}] = \text{production}$$

$$\frac{2}{3}\delta_{ij}\varepsilon = \text{dissipation,}$$

$$c_1\frac{\varepsilon}{k}(\overline{u'_i u'_j} - \frac{2}{3}\delta_{ij}k) = \text{pressure-strain}$$

$$c_s\frac{\partial}{\partial x_k}\{\frac{k}{\varepsilon}[\overline{u'_i u'_l}\frac{\partial\overline{u'_j u'_k}}{\partial x_l} + \overline{u'_j u'_l}\frac{\partial\overline{u'_k u'_i}}{\partial x_l} + \overline{u'_k u'_l}\frac{\partial\overline{u'_i u'_j}}{\partial x_l}]\} = \text{diffusion.}$$

Dissipation Equation

$$\underbrace{(\frac{\partial}{\partial t}+U_k\frac{\partial}{\partial x_k})\varepsilon}_{\text{Convection}}=\underbrace{c_\varepsilon\frac{\partial}{\partial x_k}(\frac{k}{\varepsilon}\overline{u'_k u'_i}\frac{\partial\varepsilon}{\partial x_i})}_{\text{Diffusion}}-\underbrace{c_{\varepsilon 1}\frac{\varepsilon}{k}\overline{u'_i u'_k}\frac{\partial U_i}{\partial x_k}}_{\text{Generation}}-\underbrace{c_{\varepsilon 2}\frac{\varepsilon^2}{k}}_{\text{Destruction}},$$

where

$$c_\varepsilon\frac{\partial}{\partial x_k}(\frac{k}{\varepsilon}\overline{u'_k u'_l}\frac{\partial\varepsilon}{\partial x_k}) = \text{diffusion},$$

$$c_{\varepsilon 1}\frac{\varepsilon}{k}\overline{u'_i u'_k}\frac{\partial U_i}{\partial x_k} = \text{generation}$$

$$c_{\varepsilon 2}\frac{\varepsilon^2}{k} = \text{destruction.}$$

Reynolds Equation

$$(\frac{\partial}{\partial t}+U_j\frac{\partial}{\partial x_j})U_i=-\frac{1}{\rho}\frac{\partial P}{\partial x_i}-\frac{\partial}{\partial x_j}\overline{u'_i u'_j}$$

Continuity Equation

$$\frac{\partial U_i}{\partial x_i}=0$$

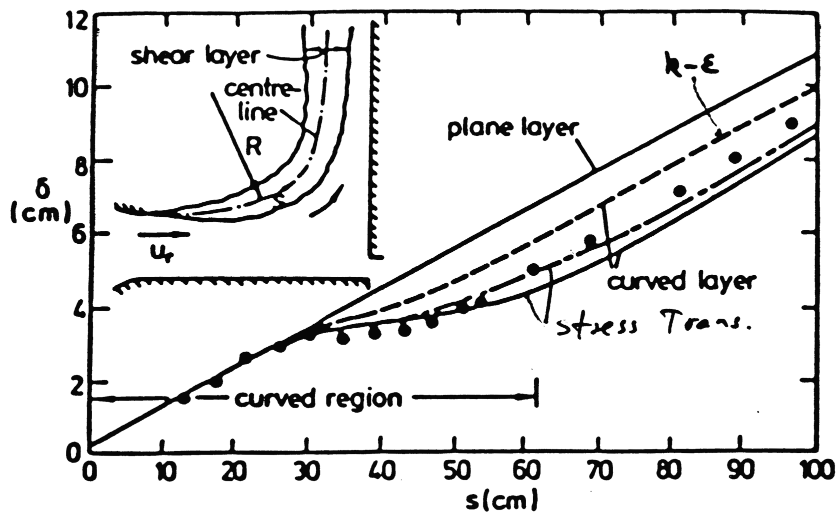
Closure

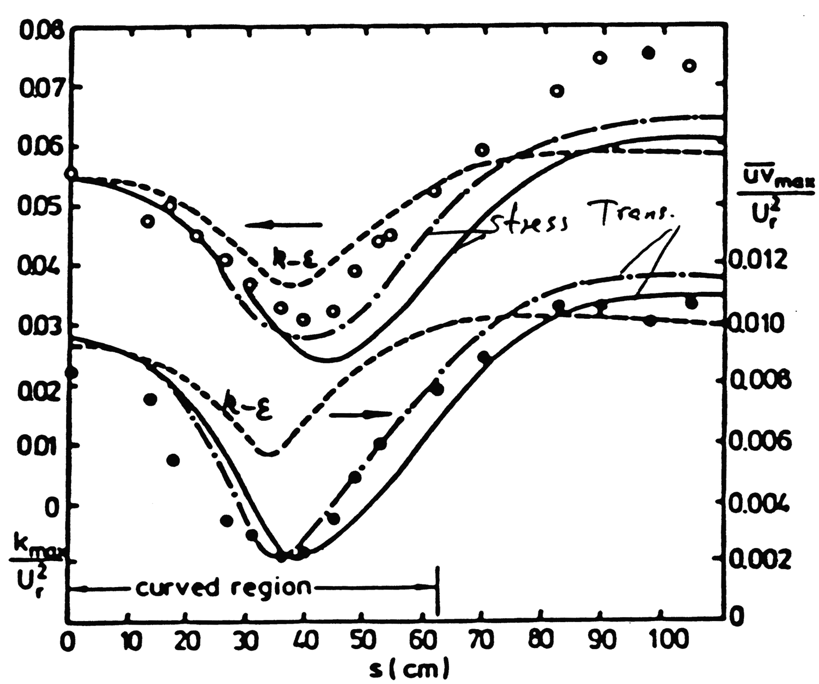
There are eleven equations for eleven unknowns, U_i, P, $\overline{u'_i u'_j}$, ε.

Comparison of Model Predictions

In this section comparisons of the predictions of the stress transport model with the experimental data for simple turbulent shear flows are presented.

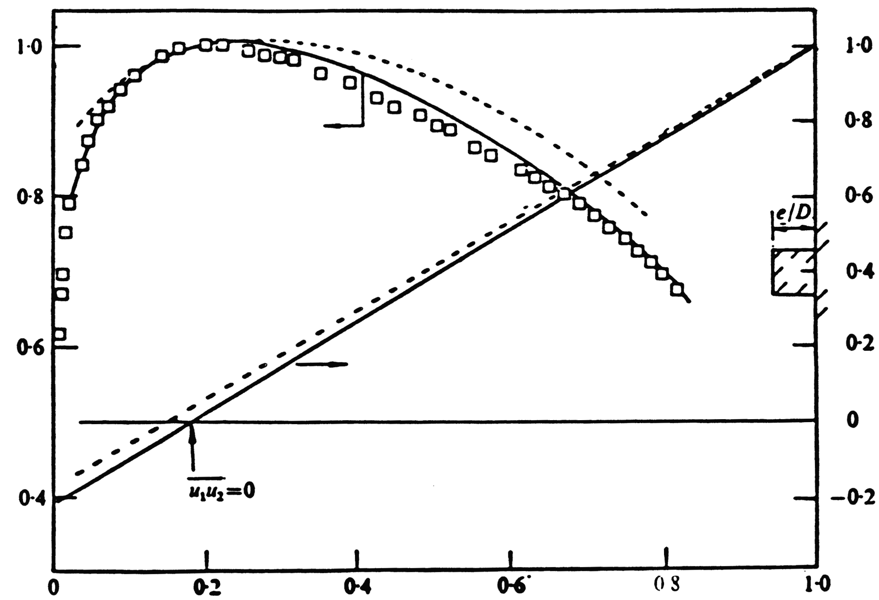
Curved Mixing Layer (Gibson and Rodi, 1981)

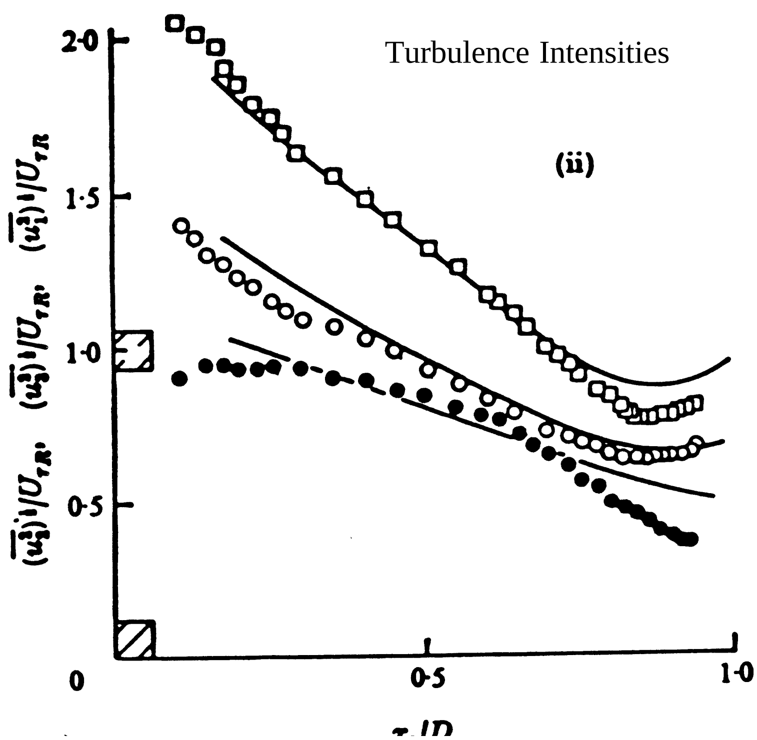




Asymmetric Channel Flow (Launder, Reece and Rodi, 1975)

Mean Velocity and Turbulent Shear Stress





Turbulence Intensities

Algebraic Stress Transport Model (Rodi, ZAMM 56, 1976)

A simplified stress transport model is given as:

$$\frac{d}{dt}\overline{u_i'u_j'} = \underbrace{c_s\frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_m'}\frac{\partial}{\partial x_m}\overline{u_i'u_j'}\right)}_{\text{Diffusion}} - \underbrace{\overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k}}_{\text{Production}} \underbrace{- c_1\frac{\varepsilon}{k}\left(\overline{u_i'u_j'} - \delta_{ij}\frac{2}{3}k\right) - \gamma\left(P_{ij} - \delta_{ij}\frac{2}{3}P\right)}_{\text{Pressure-Strain}} \underbrace{- \frac{2}{3}\delta_{ij}\varepsilon}_{\text{Dissipation}}, \tag{1}$$

where

$$D_{ij} = \frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_l'}\frac{\partial}{\partial x_l}\overline{u_i'u_j'}\right) = \text{diffusion},$$

$$P_{ij} = \overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k} = \text{production},$$

$$c_1\frac{\varepsilon}{k}\left(\overline{u_i'u_j'} - \delta_{ij}\frac{2}{3}k\right) - \gamma\left(P_{ij} - \delta_{ij}\frac{2}{3}P\right) = \text{pressure-strain},$$

$$\frac{2}{3}\delta_{ij}\varepsilon = \text{dissipation}.$$

Here, $P = \frac{1}{2}P_{ii}$ is the production rate of turbulent kinetic energy. Contracting Equation (1), we find the transport equation for k:

$$\frac{dk}{dt} = \underbrace{c_s\frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_m'}\frac{\partial k}{\partial x_m}\right)}_{\text{Diffusion}} - \underbrace{\overline{u_k'u_m'}\frac{\partial U_k}{\partial x_m}}_{\text{Production}} \underbrace{-\varepsilon}_{\text{Dissipation}}, \tag{2}$$

where

$$D = \frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_l'}\frac{\partial k}{\partial x_l}\right) = \text{diffusion}$$

$$P = \overline{u_k'u_l'}\frac{\partial U_k}{\partial x_l} = \text{production}$$

Rodi (1976) assumed that

$$\frac{d}{dt}\overline{u_i'u_j'} - D_{ij} = \frac{\overline{u_i'u_j'}}{k}\left(\frac{dk}{dt} - D\right) = \frac{\overline{u_i'u_j'}}{k}(P - \varepsilon), \tag{3}$$

Using (3) in (1) and rearranging, the result is:

$$\overline{u_i' u_j'} = k\left[\frac{2}{3}\delta_{ij} + \frac{1-\gamma}{c_1}\frac{\frac{P_{ij}}{\varepsilon} - \frac{2}{3}\frac{P}{\varepsilon}\delta_{ij}}{1+\frac{1}{c_1}(\frac{P}{\varepsilon}-1)}\right]. \tag{4}$$

Equation (4) provides an algebraic expression for $\overline{u_i' u_j'}$.

For simple shear flows, it may be shown that equation (4) reduces to the Kolmogorov-Prandtl hypothesis with

$$\nu_T = c_\mu \frac{k^2}{\varepsilon} \tag{5}$$

and

$$c_\mu = \frac{2}{3}\frac{(1-\gamma)}{c_1}\frac{[1-\frac{1}{c_1}(1-\gamma\frac{P}{\varepsilon})]}{[1+\frac{1}{c_1}(\frac{P}{\varepsilon}-1)]^2} \quad \text{with } g = 0.6 \text{ and } c_1 = 1.8 - 2.2\,. \tag{6}$$

Conclusions (Existing Models)

- Available models can predict the mean flow properties with reasonable accuracy. Small adjustments of parameters are sometimes necessary!
- First-order modeling gives reasonable results only when a single length and velocity scale characterizes turbulence.
- The $k-\varepsilon$ model gives relatively accurate results when a scalar eddy viscosity is sufficient to characterize the flow. That is there is no preferred direction for example through the action of a body force.
- The stress transport models have the potential to most accurately represent the mean turbulent flow fields.

Deficiencies of Existing Models

- Adjustments of coefficients are sometimes needed.
- The derivation of the models are somewhat arbitrary.
- There is no systematic method for improving a model when it loses its accuracy.
- Models for complicated turbulent flows (such as multiphase flows) are not available.
- Realizability and other fundamental principles are sometimes violated.

For example, the transport equations for $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$ must always lead to positive values of these quantities. In addition, the transport equations for the cross terms must also lead to cross correlations that satisfy Schwarts inequalities. i.e., $\overline{u'^2}\,\overline{v'^2} - \overline{u'v'}^2 \geq 0$.

Anisotropic Rate-Dependent Model

Averaged Balance Laws:

Mass

$$v_{i,i} = 0$$

Linear Momentum

$$\rho \dot{v}_i = t_{ji,j} + t^T_{ji,j} + \rho f_i$$

Thermal Energy

$$\rho \dot{e} = q_{i,i} + q^T_{i,i} + t_{ij} v_{j,i} + \rho \varepsilon + r$$

Fluctuation Energy

$$\rho \dot{k} = t^T_{ij} v_{j,i} + K_{i,i} - \rho \varepsilon$$

Clausius-Duhem Inequality

$$\rho \dot{\eta} - (q_i \vartheta)_{,i} - R^T_{i,i} - r\vartheta + \rho \dot{\eta}^T - S^T_{i,i} \geq 0$$

Helmholtz free Energy Function

$$\psi = e - \frac{\eta}{\vartheta}, \quad \psi^T = k - \frac{\eta^T}{\vartheta^T}$$

Heat Flux-Coldness Correlation

$$R^T_i = q^T_i \vartheta$$

Fluctuation Energy Flux-Turbulence Coldness Correlation

$$S^T_i = K_i \vartheta^T - E_i$$

Total Heat Flux

$$Q_i = q_i + q^T_i$$

Clausius-Duhem Inequality

$$\vartheta\left[-\rho\left(\dot{\psi}-\frac{\eta\dot{\vartheta}}{\vartheta^2}\right)-\frac{1}{\vartheta}Q_i\vartheta_{,i}+t_{ij}v_{j,i}+\rho\varepsilon\right]$$
$$+\vartheta^T\left\{-\rho\left[\dot{\psi}^T-\frac{\eta^T\dot{\vartheta}^T}{(\vartheta^T)^2}\right]-\frac{1}{\vartheta^T}K_i\vartheta^T_{,i}+\frac{1}{\vartheta^T E_{i,i}}+t^T_{ij}v_{j,i}-\rho\varepsilon\right\}\geq 0$$

Constitutive Equations

Stress

$$t_{ij}=-p\delta_{ij}+2\mu d_{ij}$$

$$t^T_{ij}=-\frac{2}{3}\rho k\delta_{ij}+\rho\frac{\partial\psi^T}{\partial\Delta}\frac{\hat{D}d_{ij}}{Dt}+\mu^T\left\{\left(2+\gamma\tau^2 d_{kl}d_{kl}\right)d_{ij}+\beta\tau\left[\frac{1}{3}d_{kl}d_{kl}\delta_{ij}-d_{ik}d_{kj}\right]\right\}$$

Jaumann Derivative

$$\frac{\hat{D}d_{ij}}{Dt}=\dot{d}_{ij}+d_{ik}\omega_{kj}+d_{jk}\omega_{ki}$$

Deformation Rate and Spin Tensors

$$d_{ij}=\frac{1}{2}\left(v_{i,j}+v_{j,i}\right),\qquad \omega_{ij}=\frac{1}{2}(v_{i,j}-v_{j,i}),\qquad \Delta=\frac{1}{2}d_{ij}d_{ij}$$

Heat Flux

$$Q_i=\left(\kappa+C\frac{\mu^T}{\sigma^\theta}\right)\theta_{,i}$$

Fluctuation Energy flux

$$K_i=\left(\mu+\frac{\mu^T}{\sigma^k}\right)\left[k_{,i}-\frac{k}{\tau}\tau_{,i}\right]$$

Heat Capacity

$$C=-\theta\frac{\partial^2\psi}{\partial\theta^2}$$

Thermodynamic Constraints

$$\mu^T \geq 0,\ \sigma^\theta \geq 0,\ \gamma \geq 0,\ |b|^2 \leq 48g$$

$$\mu^T = C^\mu \rho k \tau$$

Turbulence free Energy Function

$$\psi^T = k\left[\ln\left(\frac{\tau^{\alpha_o}}{k}\right) + \alpha C^\mu \tau^2 \Delta + C_0\right]$$

Turbulence Stress

$$t_{ij}^T = -\frac{2}{3}\rho k \delta_{ij} + \mu^T\left\{2d_{ij} + \alpha\tau\frac{\hat{D}}{Dt}d_{ij} + \gamma\tau^2 d_{lk}d_{kl}d_{ij} + \beta\tau\left[\frac{1}{3}d_{lk}d_{kl}\delta_{ij} - d_{ik}d_{kj}\right]\right\}$$

Basic Equations

$$v_{i,i} = 0$$

$$\rho\dot{v}_i = -\left[p + \frac{2}{3}\rho k\right]_{,i} + \left\{2(\mu + \mu^T)d_{ij} + \mu^T[\alpha\tau\frac{\hat{D}d_{ij}}{Dt} + \beta\tau(\frac{1}{3}d_{lk}d_{kl}\delta_{ij} - d_{ik}d_{kj}) + \gamma\tau^2 d_{lk}d_{kl}d_{ij}]\right\}_{,j} + \rho f_i$$

$$\rho C\dot{\theta} = \left[(\kappa + C\frac{\mu^T}{\sigma^\theta})\theta_{,i}\right]_{,i} + 2\mu d_{ij}d_{ij} + \rho\varepsilon + r$$

$$\rho\dot{k} = \left[(\mu + \frac{\mu^T}{\sigma^k})(k_{,i} - \frac{k}{\tau}\tau_{,i})\right]_{,i} + P + \alpha\tau\mu^T\frac{\hat{D}d_{ij}}{Dt}d_{ij} - \rho\varepsilon$$

$$P = \mu^T[2d_{ij}d_{ij} - \beta\tau d_{ik}d_{kj}d_{ij} + \gamma\tau^2(d_{ij}d_{ji})^2]$$

Scale Transport Equations

$$\rho\dot{\tau} = \left[(\mu + \frac{\mu^T}{\sigma^T})\tau_{,i}\right]_{,i} + C^{\tau_1}\frac{\tau}{k}P + C^{\tau_3}\left(\mu + \frac{\mu^T}{\sigma^k}\right)\left(\frac{\tau}{k^2}\right)[k_{,i} - \frac{k}{\tau}\tau_{,i}][k_{,i} - \frac{k}{\tau}\tau_{,i}]$$

$$+\left(\mu + \frac{\mu^T}{\sigma^T}\right)[\frac{2\alpha C^\mu}{\alpha_0 + 2\alpha C^\mu \tau^2 \Delta}](\tau^2\Delta)_{,i}\tau_{,i} - \rho C^{\tau_2}C^D$$

$$\frac{1}{\alpha_{0m}} \geq C^{\tau_1} \geq 0,\ C^{\tau_2} \geq \frac{1}{\alpha_0},\ \frac{1}{\alpha_{0m}} \geq C^{\tau_3} \geq 0,\ \alpha_0 \geq 0$$

$$\alpha_{0m}\ \max.(\alpha_0 + 2\alpha C^\mu \tau^2 \Delta)$$

$$\varepsilon = C^D\frac{k}{\tau}$$

$$\rho\dot{\varepsilon} = \left[(\mu + \frac{\mu^T}{\sigma^\varepsilon})\varepsilon_{,i}\right]_{,i} + C^{\varepsilon_1}\frac{\varepsilon}{k}P + C^{\varepsilon_3}\left(\mu + \frac{\mu^T}{\sigma^k}\right)\left(\frac{\varepsilon}{k^2}\right)[k_{,i} - \frac{k}{\varepsilon}\varepsilon_{,i}][k_{,i} - \frac{k}{\varepsilon}\varepsilon_{,i}]$$

$$+\left(\mu + \frac{\mu^T}{\sigma^\varepsilon}\right)[\frac{2\alpha C^\mu}{\alpha_0 + 2\alpha C^\mu \Delta\frac{k^2}{\varepsilon^2}}](\Delta\frac{k^2}{\varepsilon^2})_{,i}\varepsilon_{,i} - \rho C^{\varepsilon 2}C^{\varepsilon_2}\frac{\varepsilon^2}{k}$$

$$\mu^T = \rho C^\mu \frac{k^2}{\varepsilon},\ \tau = \frac{k}{\varepsilon}$$

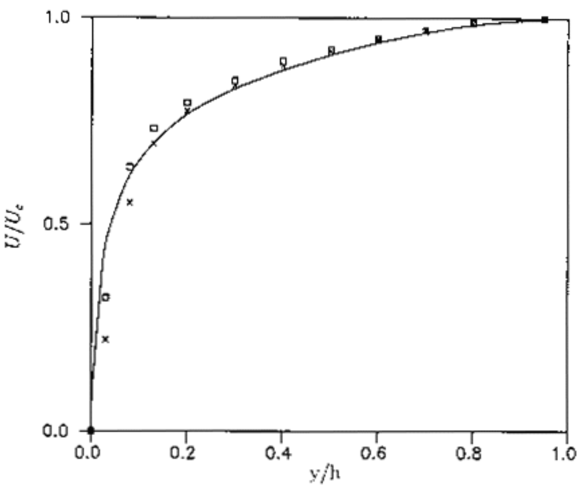
When $\gamma = 0,\ \alpha = 0.93,\ \beta = 0.54$

$$g \geq \frac{b^2}{48},\ g = 0.005$$

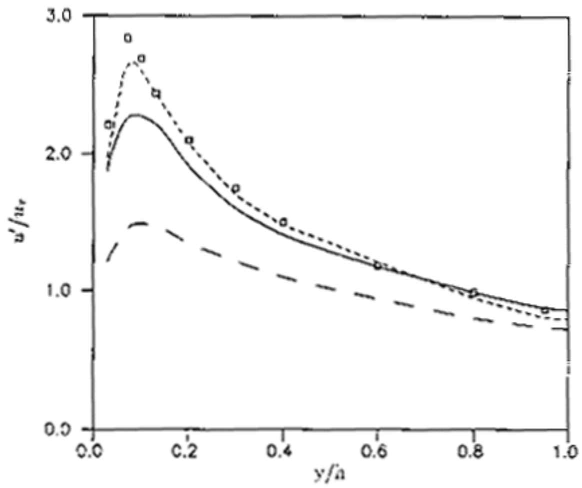
$$C^\mu = 0.09,\ C^{\varepsilon 1} = 1.45,\ C^{\varepsilon 2} = 1.92,\ \sigma^k = 1,\ \sigma^\varepsilon = 1.3$$

Comparison with the Experimental Data for Duct Flow

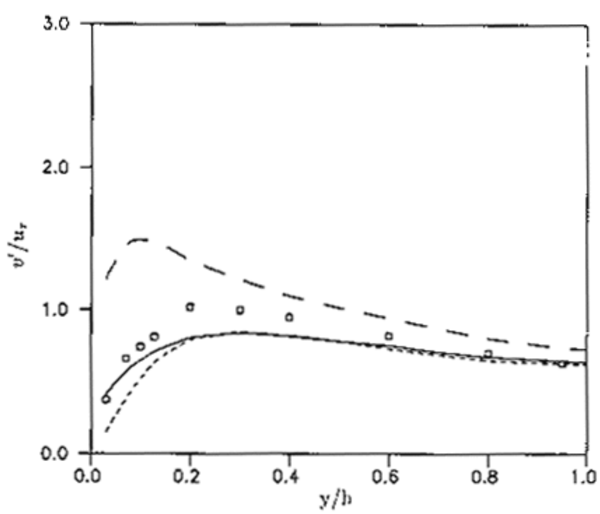
In this section the rate-dependent model predictions are compared with the experimental data of Kreplin and Eckelmann, DNS of Kim et al. and the k-ε model predictions



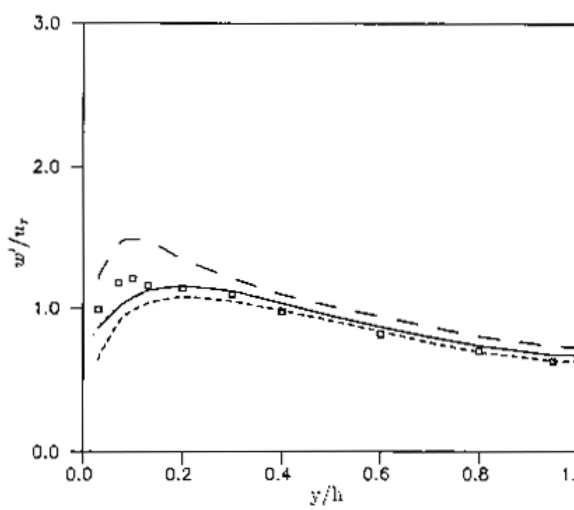
Comparison of mean velocity profile with the experimental data of Kreplin and Eckelmann for Reynolds numbers of 8200 and 5600.



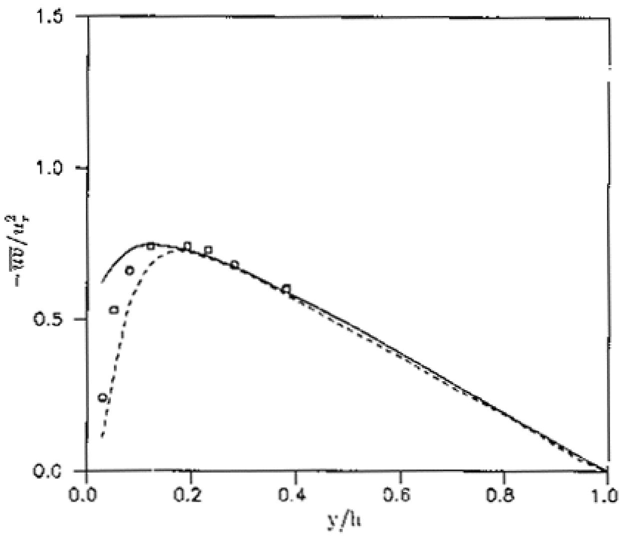
Comparison of axial turbulence intensity with the experimental data of Kreplin and Eckelmann, DNS of Kim et al. and k-ε model.



Comparison of vertical turbulence intensity with the experimental data of Kreplin and Eckelmann, DNS of Kim et al. and k-ε model.



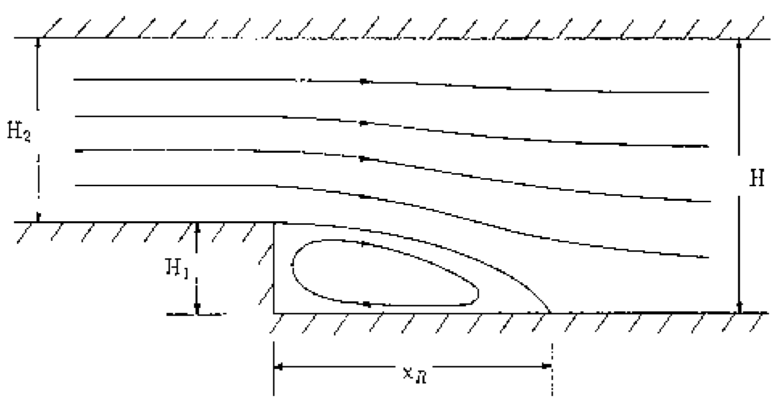
Comparison of lateral turbulence intensity with the experimental data of Kreplin and Eckelmann, DNS of Kim et al. and k-ε model.



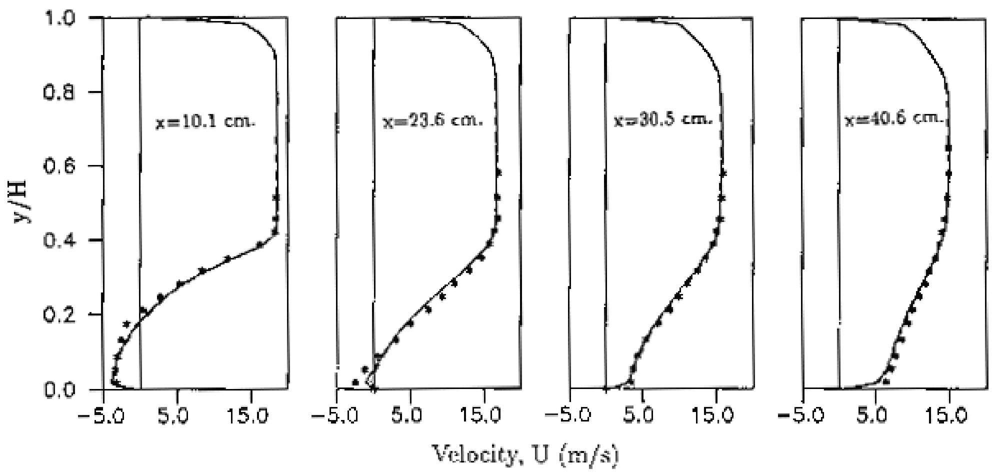
Comparison of turbulence shear stress with the experimental data of Kreplin and Eckelmann, and DNS of Kim et al.

Comparison with Experimental the Data for Backward Facing Step Flows

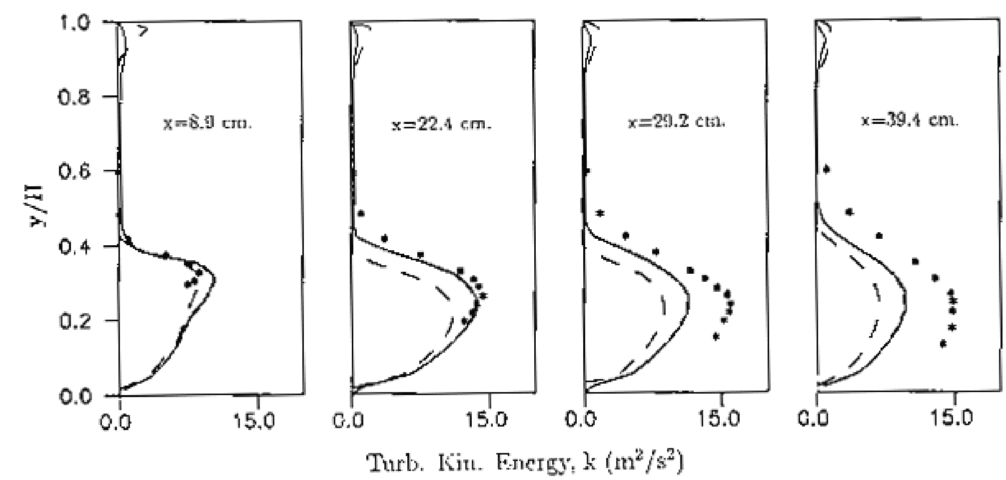
In the section the rate-dependent model predictions are compared with the experimental data of Kim et al. and the algebraic model of Srinivasan et al.



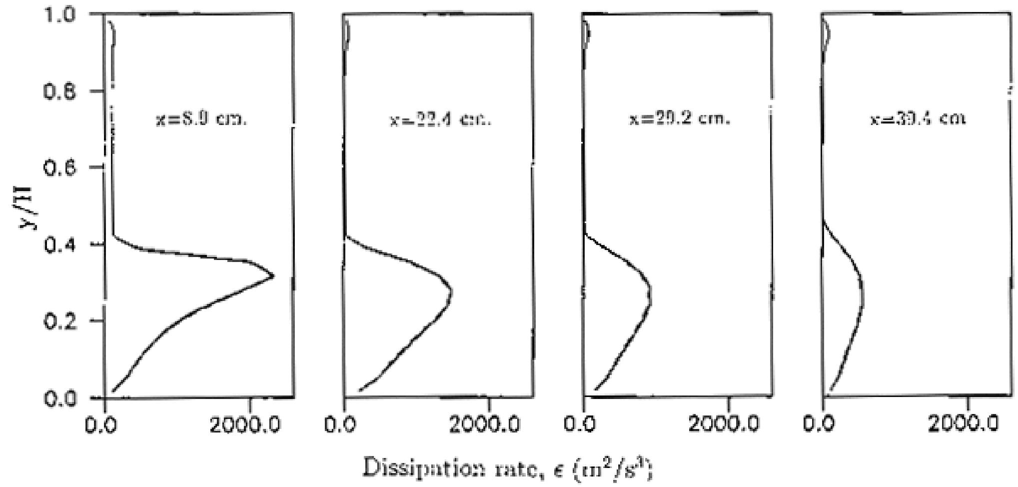
Schematics of the flow over a backward facing step.



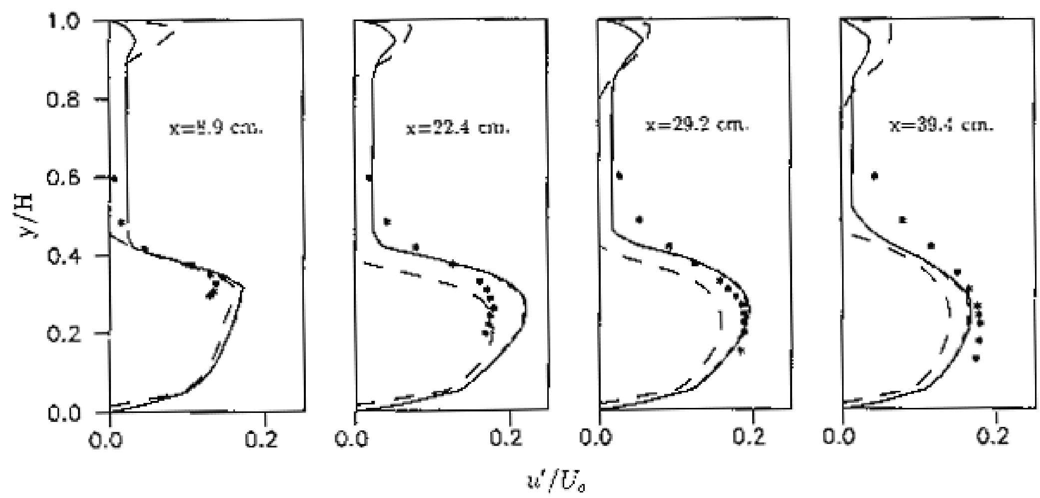
Comparison of the mean velocity profiles with the data of Kim et al. (1978).
(Dashed lines are the model predictions of Srinivasan et al. (1983).



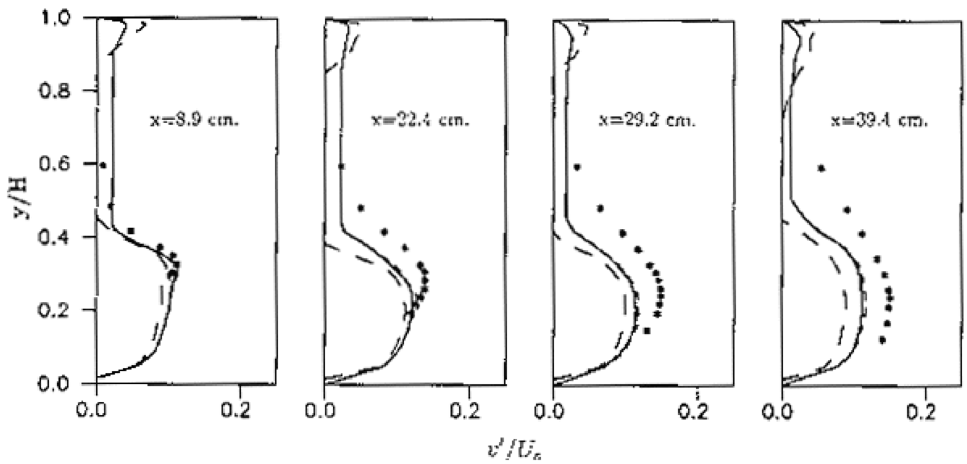
Comparison of the turbulence kinetic energy profiles with the data of Kim et al. (1978).
(Dashed lines are the model predictions of Srinivasan et al. (1983).



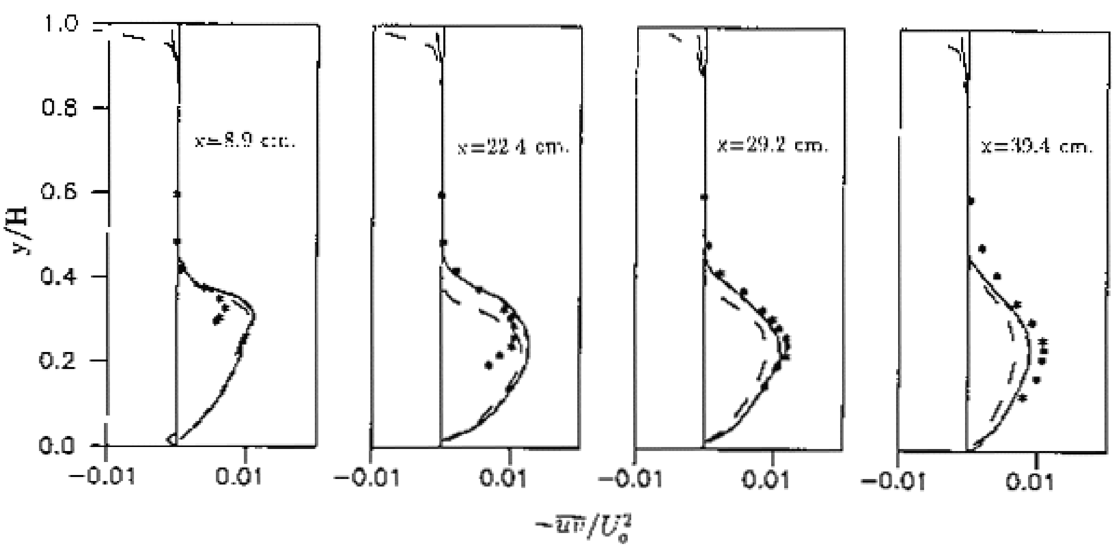
Comparison of the turbulence dissipation profiles.
(Dashed lines are the model predictions of Srinivasan et al. (1983).



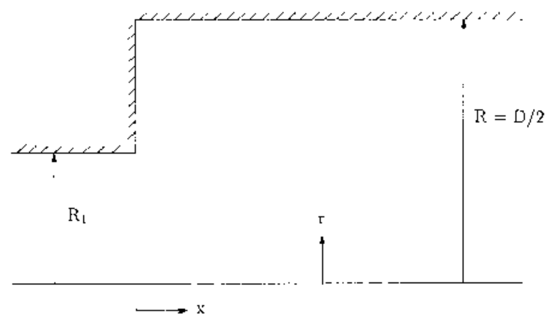
Comparison of the axial turbulence intensity profiles with the data of Kim et al. (1978).
(Dashed lines are the model predictions of Srinivasan et al. (1983).



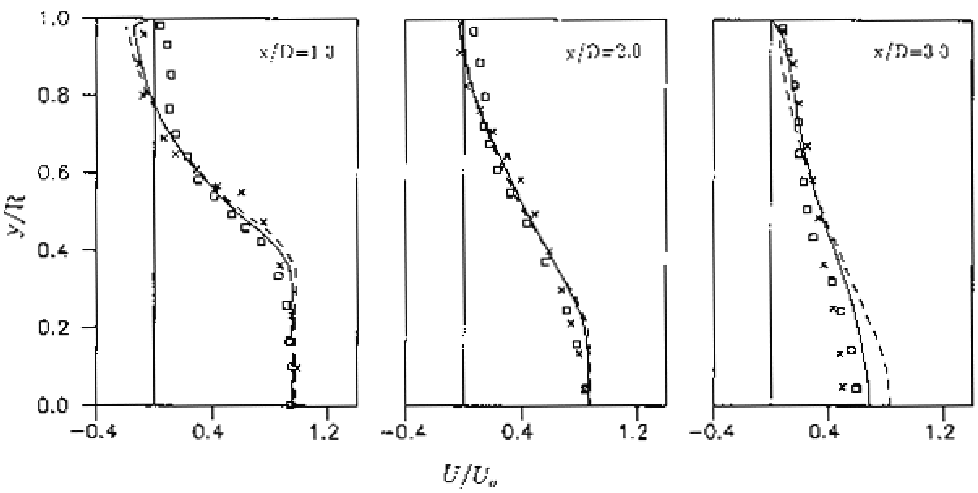
Comparison of the vertical turbulence intensity profiles with the data of Kim et al. (1978). (Dashed lines are the model predictions of Srinivasan et al. (1983).



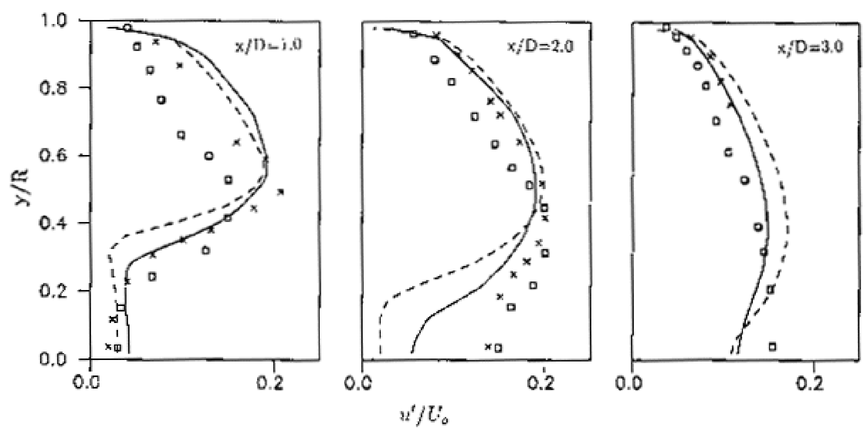
Comparison of the turbulence shear stress profiles with the data of Kim et al. (1978). (Dashed lines are the model predictions of Srinivasan et al. (1983).



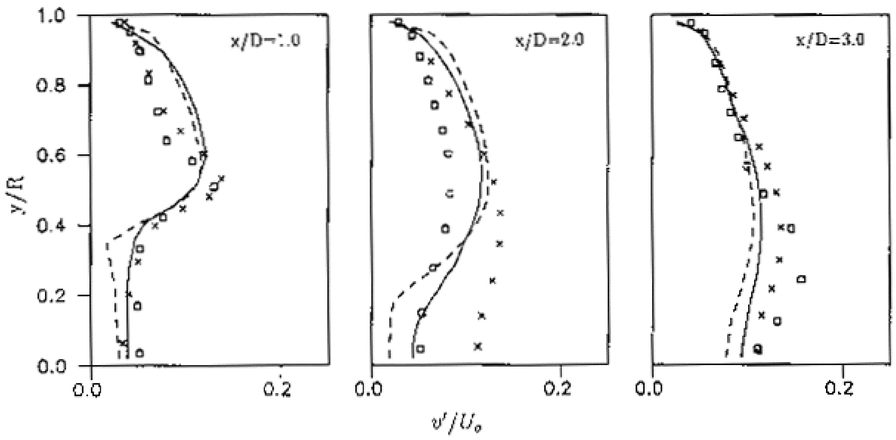
Schematics of the flow in an axisymmetric pipe expansion.



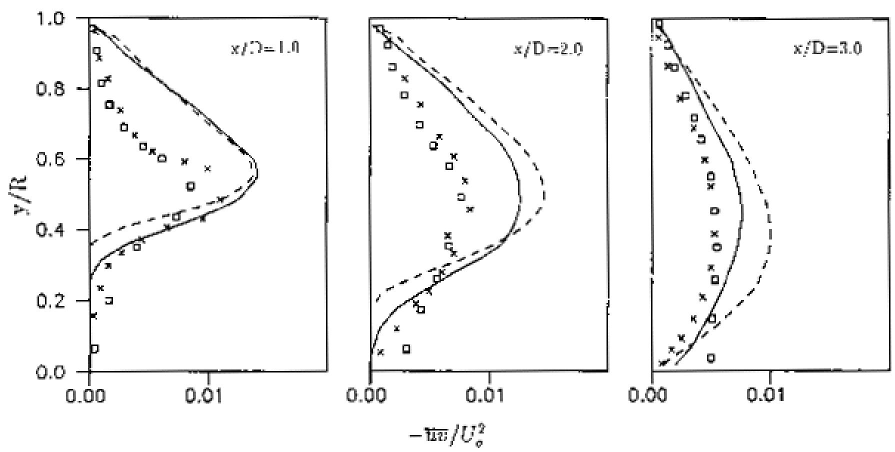
Comparison of the mean velocity profiles with the data of Junjua et al. (1982) and Chaturvedi (1963). (Dashed lines are the model predictions of Srinivasan et al. (1983))



Comparison of the axial turbulence intensity profiles with the data of Junjua et al. (1982) and Chaturvedi (1963). (Dashed lines are the model predictions of Srinivasan et al. (1983))



Comparison of the vertical turbulence intensity profiles with the data of Junjua et al. (1982) and Chaturvedi (1963). (Dashed lines are the model predictions of Srinivasan et al. (1983))



Comparison of the turbulence shear stress profiles with the data of Junjua et al. (1982) and Chaturvedi (1963). (Dashed lines are the model predictions of Srinivasan et al. (1983))