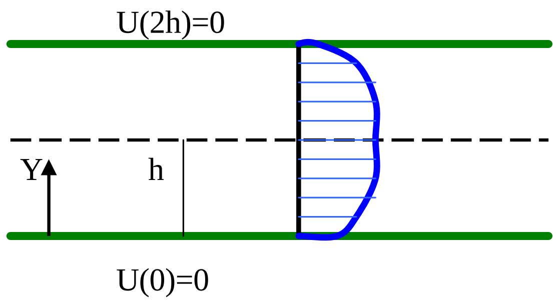
Turbulent Flow Between Two Parallel Plates

Consider a turbulent flow field between two parallel plates as shown in the figure.



The Reynolds Equation for the mean turbulent motion is given by

$$U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} \tag{1}$$

Mean Flow Equations

Let

$$\mathbf{U} = (U(y), 0, 0) \tag{2}$$

Equation (1) leads to

x-comp:
$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{d}{dy}\overline{u'v'} + \nu \frac{d^2 U}{dy^2} \tag{3}$$

y-comp:
$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} - \frac{d}{dy}\overline{v'^2} \tag{4}$$

Integrating Equation (4), we find

$$\frac{P}{\rho} + \overline{v'^2} = \frac{P_0}{\rho}, \quad \text{or} \quad \frac{\partial P}{\partial x} = \frac{dP_0}{dx}. \tag{5}$$

Equation (3) now becomes

$$0 = -\frac{1}{\rho}\frac{dP_0}{dx} - \frac{d}{dy}\overline{u'v'} + \nu \frac{d^2 U}{dy^2}. \tag{6}$$

Integrating (6) and noting that $\nu \left.\frac{dU}{dy}\right|_{y=0} = \frac{\tau_0}{\rho} = u^{*2}$, we find

$$-\frac{y}{\rho}\frac{dP_0}{dx} - \overline{u'v'} + \nu\frac{dU}{dy} - u^{*2} = 0 . \tag{7}$$

At the centerline $y = h$, equation (7) implies

$$-\frac{h}{\rho}\frac{dP_0}{dx} = u^{*2} . \tag{8}$$

Eliminating $\frac{dP_0}{dx}$ between (7) and (8), the result is

$$-\overline{u'v'} + \nu\frac{dU}{dy} = u^{*2}\left(1 - \frac{y}{h}\right) \tag{9}$$

Introducing the dimensionless variable $\eta = \frac{y}{h}$, equation (9) may be restated as

$$-\frac{\overline{u'v'}}{u^{*2}} + \frac{1}{R^*}\frac{d}{d\eta}(U^+) = 1 - \eta , \tag{10}$$

where

$$U^+ = \frac{U}{u^*} \text{ and } R^* = \frac{u^* h}{\nu} . \tag{11}$$

Alternatively introducing $y^+ = \frac{yu^*}{\nu}$, equation (9) becomes

$$-\frac{\overline{u'v'}}{u^{*2}} + \frac{dU^+}{dy^+} = 1 - \frac{1}{R^*} y^+ . \tag{12}$$

For high Reynolds number flows as $R^* \to \infty$, equations (10) and (12) imply

$$-\frac{\overline{u'v'}}{u^{*2}} = 1 - \eta \qquad (\text{as } R^* \to \infty,\ \eta \sim 1 \text{ (core region)}) \tag{13}$$

$$-\frac{\overline{u'v'}}{u^{*2}}+\frac{dU^+}{dy^+}=1 \qquad (\text{as } R^* \to \infty,\ y^+ \sim 1 \text{ (surface layer)}) \tag{14}$$

Law of the Wall

We expect the solution to (14) to be given as

$$\text{Law of the Wall:} \qquad \left\{ \begin{array}{l} U^+ = f(y^+) \\ -\dfrac{\overline{u'v'}}{u^{*2}} = g(y^+) \end{array} \right\}, \tag{15}$$

with boundary conditions $f(0)=0$ and $g(0)=0$, and the shapes of $f(y^+)$ and $g(y^+)$ are to be found experimentally.

Velocity Defect Law

In the core region, the turbulent stresses are given by (13) and the mean velocity is given as

$$\text{Velocity Defect Law:} \qquad \frac{U-U_0}{u^*}=F(\eta) \tag{16}$$

The velocity gradients from (15) and (16) may be found, i.e.

$$\frac{dU}{dy}=\frac{u^{*2}}{\nu}\frac{df}{dy^+} \tag{17}$$

and

$$\frac{dU}{dy}=\frac{u^*}{h}\frac{dF}{d\eta}. \tag{18}$$

Inertial Sublayer

From equations (17) and (18) in the limit of $\eta << 1$ and $y^+ >> 1$, we find

$$\frac{dU}{dy}=\frac{u^{*2}}{\nu}\frac{df}{dy^+}=\frac{u^*}{h}\frac{dF}{d\eta} \qquad (\text{as } \eta \to 0,\ y^+ \to \infty) \tag{19}$$

Multiplying (19) by $\frac{y}{u^*}$, the result is

$$y^{+}\frac{df(y^{+})}{dy^{+}}=\eta\frac{dF(\eta)}{d\eta}=\frac{1}{\kappa}=\text{const.} \tag{20}$$

Solving, we find

$$F(\eta)=\frac{1}{\kappa}\ln\eta+c_{1} \qquad \text{for } \eta<<1 \tag{21}$$

and

$$f(y^{+})=\frac{1}{\kappa}\ln y^{+}+c_{2} \quad \text{for } y^{+}>>1 \tag{22}$$

In the inertial sublayer from (13) or (14), we conclude that

$$-\frac{\overline{u'v'}}{u^{*2}}=1. \tag{23}$$

Logarithmic Friction Law

The velocity defect law and the law of the wall in the inertial sublayer are given as

$$\frac{U-U_{0}}{u^{*}}=\frac{1}{\kappa}\ln\eta+c_{1}, \tag{24}$$

$$\frac{U}{u^{*}}=\frac{1}{\kappa}\ln y^{+}+c_{2}\ . \tag{25}$$

Subtracting, we find

$$\frac{U_{0}}{u^{*}}=\frac{1}{\kappa}\ln R^{*}+c_{2}-c_{1} \qquad (R^{*}=\frac{u^{*}h}{\nu}) \tag{26}$$

with c_1 and c_2 known, equation (26) is the statement of the logarithmic friction law.

Pipe Flow

The law of the wall and the velocity defect law are also valid for turbulent pipe flows. Equations (9) - (26) can be written for pipe flows with the following minor changes:

$$\eta = \frac{y}{r_0} \text{ and } R^* = \frac{u^* r_0}{\nu}. \tag{27}$$

Here, r_0 is the radius of the pipe and y is the distance from the wall. For pipe flows, $\kappa = 0.4$ and equations (24) - (26) become

$$U^+ = \frac{U}{U^*} = 2.5 \ln y^+ + 5, \text{ is valid for up to } \eta \approx 0.25 \tag{28}$$

$$\frac{U - U_0}{u^*} = 2.5 \ln \eta - 1 \tag{29}$$

$$\frac{U_0}{u^*} = 2.5 R^* + 6 \tag{30}$$

Wake Function

The wake function is defined as

Law of the Wake: $$W(\eta) = 1 - 2.5 \ln \eta + F(\eta), \tag{31}$$

where $F(\eta)$ is the velocity defect law. Experiment shows that

$$W(\eta) = \frac{1}{2}\left[\sin \pi(\eta - \frac{1}{2}) + 1\right]. \tag{32}$$

Viscous Sublayer

In the viscous sublayer, the Reynolds stress is negligible. Equation (14) then becomes

$$\frac{dU^+}{dy^+} = 1. \tag{33}$$

or

$$U^+ = y^+ \tag{34}$$

Kolmogorov Scale

In the inertial sublayer $-\overline{u'v'} \approx u^{*2}$ and $\frac{\partial U}{\partial y} \approx \frac{u^*}{\kappa y}$. The turbulent production then is given as

$$\text{Production} = -\overline{u'v'}\frac{\partial U}{\partial y} = \frac{u^{*3}}{\kappa y}. \tag{35}$$

Experiment shows that in the inertial sublayer, production is equal to dissipation:

$$\varepsilon = \frac{u^{*3}}{\kappa y} \tag{36}$$

Kolmogorov scale η is given by

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} \tag{37}$$

Let

$$\eta^+ = \frac{\eta u^*}{\nu}, \tag{37}$$

then

$$\eta^+ = \left(\frac{u^{*4}\nu^3}{\nu^4 \frac{u^{*3}}{\kappa y}}\right)^{\frac{1}{4}} \qquad \text{or} \qquad \eta^+ = (\kappa y^+)^{\frac{1}{4}} \tag{38}$$

The turbulent macroscale near the wall is given as

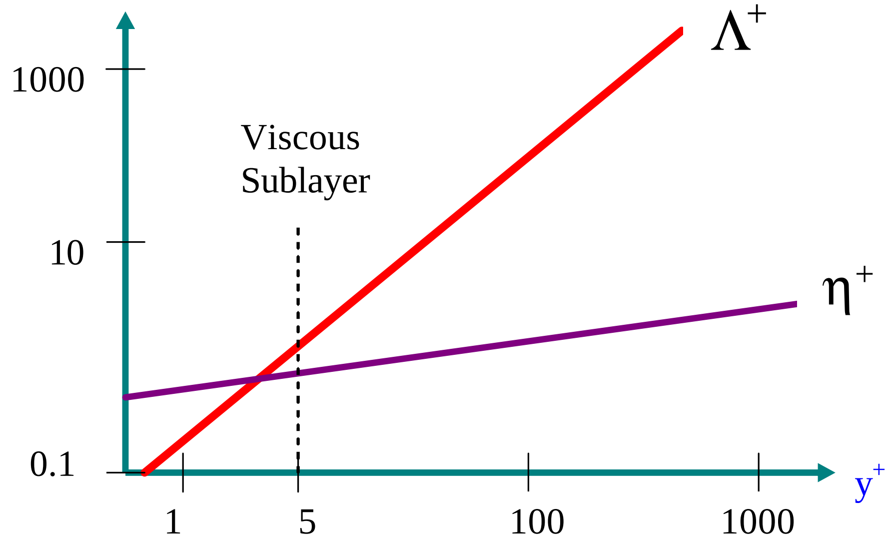
$$\Lambda = \kappa y \tag{39}$$

or

$$\Lambda^+ = \frac{\Lambda u^*}{\nu} = \kappa y^+ \tag{40}$$

Table of variation of the scales near a wall

y^+	$\eta^+ = (\kappa y^+)^{\frac{1}{4}}$	$\Lambda^+ = \kappa y^+$
5	1.2	2
12	1.5	4
40	2	16
200	3	80
1000	4.5	400



Schematic variations of the macroscale and Kolmogorov scale in turbulent near wall flows

From the table and the schematics diagram, it is observed that for $y^+ \le \frac{1}{\kappa} = 2.5$, $\Lambda^+ < \eta^+$ and a turbulent flow cannot exist.