ME 639 - Turbulence

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Turbulence

Vorticity Equation

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Vorticity Transport in Turbulent Flows

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Outline

- Vorticity Transport in Viscous Flows
- Mean Flow Vorticity Energy Budget
- Turbulence Mean Square Vorticity
- Order of Magnitude Analysis

Vorticity Transport

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Navier-Stokes Equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}$$

$$\nabla\cdot\mathbf{u}$$

Vector Identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla\frac{|\mathbf{u}|^2}{2} = \boldsymbol{\omega}\times\mathbf{u} + \nabla\frac{|\mathbf{u}|^2}{2}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} + \nabla\frac{|\mathbf{u}|^2}{2} = -\nabla\frac{p}{\rho} + \nu\nabla^2\mathbf{u}$$

Vorticity Transport Equation

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Vorticity → $\boldsymbol{\omega} = \nabla\times\mathbf{u}$

Vector Identities

$$\nabla\times\nabla\times\mathbf{u} = \nabla\nabla\cdot\mathbf{u} - \nabla^2\mathbf{u} = -\nabla^2\mathbf{u}$$

Vorticity Transport Eq. →

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\mathbf{u}) = \nu\nabla^2\boldsymbol{\omega}$$

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Vector Identities

$$\nabla\times(\boldsymbol{\omega}\times\mathbf{u})=\mathbf{u}\cdot\nabla\boldsymbol{\omega}-\boldsymbol{\omega}\cdot\nabla\mathbf{u}+\boldsymbol{\omega}(\nabla\cdot\mathbf{u})-\mathbf{u}(\nabla\cdot\boldsymbol{\omega})$$
$$=\mathbf{u}\cdot\nabla\boldsymbol{\omega}-\boldsymbol{\omega}\cdot\nabla\mathbf{u}$$

Vorticity Transport Eq.

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\boldsymbol{\omega}=\boldsymbol{\omega}\cdot\nabla\mathbf{u}+\nu\nabla^2\boldsymbol{\omega}$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\boldsymbol{\omega}=\mathbf{d}\cdot\boldsymbol{\omega}+\nu\nabla^2\boldsymbol{\omega}$$

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Turbulent Flows

$\mathbf{u}=\mathbf{U}+\mathbf{u}'$ $U_i=\overline{u_i}$ $\overline{u_i'}=0$

$\boldsymbol{\omega}=\boldsymbol{\Omega}+\boldsymbol{\omega}'$ $\boldsymbol{\Omega}=\overline{\boldsymbol{\omega}}$ $\overline{\boldsymbol{\omega}'}=0$

Mean Vorticity Transport Equation

$$\frac{\partial\Omega_i}{\partial t}+U_j\frac{\partial\Omega_i}{\partial x_j}=-\frac{\partial\overline{\omega_i'u_j'}}{\partial x_j}+\frac{\partial\overline{\omega_j'u_i'}}{\partial x_j}+\Omega_j\frac{\partial U_i}{\partial x_j}+\nu\frac{\partial^2\Omega_i}{\partial x_j\partial x_j}$$

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Instantaneous Vorticity Transport Equation

$$\frac{\partial\omega_i'}{\partial t}+U_j\frac{\partial\omega_i'}{\partial x_j}=-u_j'\frac{\partial\Omega_i}{\partial x_j}-u_j'\frac{\partial\omega_i'}{\partial x_j}+\Omega_j d_{ij}'$$
$$+\omega_j'D_{ij}+\omega_j'd_{ij}'+\nu\frac{\partial^2\omega_i'}{\partial x_j\partial x_j}+\frac{\partial\overline{\omega_i'u_j'}}{\partial x_j}-\overline{\omega_j'd_{ij}'}$$

Mean Flow Vorticity Energy Eq.

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$$\underbrace{\left(\frac{\partial}{\partial t}+\overset{u^3/\Lambda^3}{U_j\frac{\partial}{\partial x_j}}\right)\left(\frac{1}{2}\Omega_i\Omega_i\right)}_{\text{Convective Transport of mean Vorticity}}=\underbrace{-\frac{\partial}{\partial x_j}\overset{u^3/\Lambda^3}{\left(\Omega_i\overline{\omega_i'u_j'}\right)}}_{\text{transport by turbulence velocity-vorticity interaction}}+\underbrace{\overset{u^3/\Lambda^3}{\overline{u_j'\omega_i'}}\frac{\partial\Omega_i}{\partial x_j}}_{\text{gradient production of fluctuating vorticity}}$$

$$+\underbrace{\overset{u^3/\Lambda^3}{\Omega_i\Omega_jD_{ij}}}_{\text{stretching of vorticity by mean shear flow}}+\underbrace{\Omega_i\overline{\omega_j'd_{ij}'}}_{\text{stretching of vorticity by mean shear flow}}\ u^3/\Lambda^3$$

$$\nu u^2/\Lambda^4=u^3/\Lambda^3 R_\Lambda$$

$$+\underbrace{\nu\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{2}\Omega_i\Omega_i\right)}_{\text{viscous diffusion}}-\underbrace{\nu\frac{\partial\Omega_i}{\partial x_j}\frac{\partial\Omega_i}{\partial x_j}}_{\text{dissipation of mean vorticity}}$$

Fluctuation Vorticity Energy Eq.

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$$\left(\frac{\partial}{\partial t}+\overset{u^3/\Lambda\lambda^2}{U_j\frac{\partial}{\partial x_j}}\right)\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right) = \underbrace{\overset{u^3/\Lambda^3}{-\overline{u_j'\omega_i'}\frac{\partial \Omega_i}{\partial x_j}}}_{\text{gradient production of fluctuating vorticity}} \underbrace{-\frac{1}{2}\frac{\partial}{\partial x_j}\overset{u^3/\Lambda\lambda^2}{\left(\overline{u_j'\omega_i'\omega_i'}\right)}}_{\text{diffusion of turbulence vorticity by turbulence}}$$

$$+\overset{u^3/\lambda^3}{\underbrace{\overline{\omega_i'\omega_j'd_{ij}'}}_{\text{production of turbulence vorticity by turbulent stretching}}} + \underbrace{\overline{\omega_i'\omega_j'}D_{ij}}_{\text{production of turbulence vorticity by mean shear flow stretching}}\ u^3/\Lambda^2\lambda$$

$$+\overset{u^3/\Lambda^3}{\underbrace{\Omega_j\overline{\omega_i'd_{ij}'}}_{\text{production by mixed turbulence-mean flow stretching}}} \overset{\nu u^2/\lambda^2\Lambda^2 \;=\; u^3/\Lambda^3}{\underbrace{+\nu\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right)}_{\text{diffusion by viscosity}}} - \overset{\nu u^2/\lambda^2\eta^2}{\underbrace{\nu\overline{\frac{\partial\omega_i'}{\partial x_j}\frac{\partial\omega_i'}{\partial x_j}}}_{\text{dissipation destruction}}}$$

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Order of Terms

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$$\overline{\omega_i'\omega_i'} \sim \frac{u^2}{\lambda^2} \qquad \overline{u_i'\omega_i'} \sim \frac{u^2}{\Lambda} \qquad \overline{\omega_j'd_{ij}'} \sim \frac{u^2}{\Lambda^2}$$

$$\overline{\omega_i'\omega_j'} \sim \frac{u^2}{\lambda^2}\left(a\delta_{ij}+b_{ij}\frac{\lambda}{\Lambda}+\ldots\right)$$

$$\overline{\omega_i'\omega_j'}\frac{\partial U_i}{\partial x_j} = \overline{\omega_i'\omega_j'}D_{ij} \sim \frac{u^2}{\lambda^2}\frac{\lambda}{\Lambda}\frac{u}{\Lambda} = \frac{u^3}{\lambda\Lambda^2}$$

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Concluding Remarks

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Thank you!

Questions?

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