Stability Analysis for Finite Amplitude Disturbances

Let $\mathbf{v}$, p be a basic motion of a viscous fluid in a bound region V. Let S denote the surface boundary of V. The basic flow satisfies the Navier-Stokes equation and the continuity equation. In dimensionless form these are given as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \frac{1}{Re} \nabla^2 v , \qquad \nabla \cdot \mathbf{v} = 0 , \qquad \text{in } V \tag{1}$$

$$\mathbf{v} = \mathbf{V} \qquad \text{on } S . \tag{2}$$

Consider a disturbed motion $\mathbf{v}^*$, p^*. The disturbed motion must satisfy the same equations and boundary condition. These are

$$\frac{\partial \mathbf{v}^*}{\partial t} + \mathbf{v}^* \cdot \nabla \mathbf{v}^* = -\nabla p^* + \frac{1}{Re} \nabla^2 \mathbf{v}^* , \quad \nabla \cdot \mathbf{v}^* = 0 , \qquad \text{in } V , \tag{3}$$

$$\mathbf{v}^* = \mathbf{V} \text{ on } S . \tag{4}$$

The difference motion is defined as

$$\mathbf{u} = \mathbf{v}^* - \mathbf{v} , \qquad \pi = p^* - p . \tag{5}$$

Subtracting (1) from (3) and using (5), we find

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \pi + \frac{1}{Re} \nabla^2 \mathbf{u} , \qquad \text{in } V , \tag{6}$$

$$\mathbf{u} = 0 \text{ on } S . \tag{7}$$

Equation (6) is the governing equation for the finite amplitude disturbance.

The stability may be analyzed by studying the dynamics of the kinetic energy of the difference motion, T. That is,

$$T = \frac{1}{2} \int u^2 dV , \tag{8}$$

where the integral is over the volume V unless stated otherwise.

Using (6), we find

$$\frac{dT}{dt} = \int \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} dV = \int \left[\frac{1}{Re} \mathbf{u} \cdot \nabla^2 \mathbf{u} - \mathbf{u} \cdot \nabla \pi - \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{u} \right] dV \tag{9}$$

With the help of vector identities and divergence theorem, the right hand side of (9) is simplified. Using

$$\nabla \times \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}, \tag{10}$$

and

$$\mathbf{u} \cdot \nabla \times (\nabla \times \mathbf{u}) = -\nabla \cdot (\mathbf{u} \times (\nabla \times \mathbf{u})) + (\nabla \times \mathbf{u})^2 \tag{11}$$

The first term on the right hand side of (9) may be restated as

$$\int \mathbf{u} \cdot \nabla^2 \mathbf{u} dV = \int_S \mathbf{u} \times (\nabla \times \mathbf{u}) \cdot \mathbf{dS} - \int (\nabla \times \mathbf{u})^2 dV. \tag{12}$$

The second term on the right hand side of (9) becomes

$$\int \mathbf{u} \cdot \nabla \pi dV = \int [\nabla \cdot (\pi \mathbf{u}) - \pi \nabla \cdot \mathbf{u}] dV = \int_S \pi \mathbf{u} \cdot \mathbf{dS} = 0 \ . \tag{13}$$

The last two terms in (9) also vanish identically. That is,

$$\int \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{u} dV = \int \mathbf{v} \cdot \nabla \frac{u^2}{2} dV = \int \left[\nabla \cdot \left(\mathbf{v} \frac{u^2}{2} \right) - \frac{u^2}{2} \nabla \cdot \mathbf{v} \right] dV = \int_S \mathbf{v} \frac{u^2}{2} \cdot \mathbf{dS} = 0, \tag{14}$$

$$\int \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{u} = \int \mathbf{u} \cdot \nabla \frac{u^2}{2} dV = \int_S \frac{u^2}{2} \mathbf{u} \cdot \mathbf{dS} = 0. \tag{15}$$

Using (12) – (15), equation (9) may be restated as

$$\frac{dT}{dt} = -\frac{1}{Re} \int (\nabla \times \mathbf{u})^2 dV - \int \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{u} dV. \tag{16}$$

Employing the Korn inequality,

$$\int (\nabla \times \mathbf{u}) dV \geq N \int u^2 dV, \tag{17}$$

where N is a number depending on the geometry (for spheres $N = 80$) equation (16) becomes

$$\frac{dT}{dt} \leq 2\left(-\frac{N}{Re} + \lambda \right) T. \tag{18}$$

Here, λ is the maximum eigenvalue of $(-\nabla \mathbf{v})$ or $(-\mathbf{d})$ in time period 0 to t. In deriving (18), we used the following inequality:

$$-\mathbf{u} \cdot \mathbf{d} \cdot \mathbf{u} \le \lambda u^2 . \tag{19}$$

From (18), it follows that if it follows that if $\mathrm{Re} \le \frac{N}{\lambda}$ then the kinetic energy of the difference motion decays to zero and the basic motion is stable. That is, from (18), we find

$$T \le T(0) \exp\left\{-\left(\frac{N}{\mathrm{Re}} - \lambda\right) t\right\}. \tag{20}$$

As $t \to \infty$, then $T \to 0$ and $u = 0$ and $\mathbf{v}^* = \mathbf{v}$ almost everywhere. Based on these results, the following theorem regarding the stability of basic motion may be stated.

Theorem

If for a basic flow of a viscous incompressible fluid in a bounded region of space V, $\mathrm{Re} \le \frac{N}{\lambda}$, then the basic flow is stable.

Corollary 1 (Uniqueness of Unsteady Viscous Flows)

If $\mathbf{v}$ and $\mathbf{v}^*$ are two unsteady flows of a viscous fluid in a bounded region of space V having the same velocity distribution at time $t = 0$ and on boundary of V, then they must be identical if $\mathrm{Re} \le \frac{N}{\lambda}$.

Corollary 2 (Uniqueness of Steady Viscous Flows)

If $\mathbf{v}$ and $\mathbf{v}^*$ are two steady flows of a viscous incompressible fluid in a bounded region $V(t)$ subject to the same boundary conditions, then the two motions must be identical if $\mathrm{Re} \le \frac{N}{\lambda}$.