Clarkson University

CONSERVATION LAWS

Axiom 1: Principle of Conservation of Mass

Mass is invariant under the motion. That is,

$$\frac{d}{dt}\int_v \rho dv = 0 . \tag{1}$$

Using the Reynolds transport theorem, we find

$$\frac{\partial}{\partial t}\int_v \rho dv + \int_s \rho \mathbf{v}\cdot \mathbf{ds} = 0 \qquad \text{(Global)} \tag{2}$$

or

$$\int_v \left(\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} \right) dv = 0 . \tag{3}$$

That leads to the equation of continuity

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 . \qquad \text{(Local)} \tag{4}$$

Axiom 2: Principle of Balance of Linear Momentum

The time rate of change of momentum is equal to the resultant force acting on the body. That is

$$\frac{d}{dt}\int_v \rho v_k dv = \int_v \rho f_k dv + \int_s t_k^{(n)} ds , \tag{5}$$

where f_k is the acceleration of the body force and $t_k^{(n)}$ is the surface traction force. Using the Reynolds transport equation, we find

$$\frac{\partial}{\partial t}\int_v \rho v_k dv + \int_s \rho v_k v_j ds_j = \int_v \rho f_k dv + \int_s t_k^{(n)} ds . \quad \text{(Global)} \tag{6}$$

Introducing the stress tensor $t_{\ell k}$ as

$$t_k^{(n)} = t_{\ell k} n_\ell , \; \mathbf{t}^{(n)} = \mathbf{n}\cdot\mathbf{t} , \tag{7}$$

the last term in (6) may be restated as

$$\int_s t_{\ell k} n_\ell ds = \int_v t_{\ell k,\ell} dv .$$

Using the divergence theorem in the second term of (6) or noting that $\frac{d}{dt}(\rho d\mathbf{v}) = 0$ in (5), we find

$$\int_v (\rho \frac{dv_k}{dt} - \rho f_k - t_{\ell k,\ell}) dv = 0 \ .$$

This implies that

$$\rho \frac{dv_k}{dt} = \rho f_k + t_{\ell k,\ell} . \quad \text{(Local)} \tag{8}$$

Axiom 3: Principle of Balance and Angular Momentum

Time rate of change of angular momentum about a fixed point is equal to the resultant moments about that point. That is

$$\frac{d}{dt}\int_v \rho(\sigma_k + \varepsilon_{kmj} r_m v_j) dv = \int_v \rho \varepsilon_{kmj} r_m f_j dv + \int_S \varepsilon_{kmj} r_m t_j^{(n)} ds + \int_s m_k^{(n)} ds + \int_s \rho \ell_k ds , \tag{9}$$

where σ_k is the inertial spin, r_m is the position, $m_k^{(n)}$ is the surface couple, and ℓ_k is the body couple per unit mass.

Introducing the couple stress $m_{\ell k}$ as

$$m_k^{(n)} = m_{\ell k} n_\ell , \ \mathbf{m}^{(n)} = \mathbf{n} \cdot \mathbf{m} , \tag{10}$$

and applying the divergence theorem, we find

$$\int_v \rho(\dot{\sigma}_k + \varepsilon_{kmj} r_m \dot{v}_j) dv = \int_v \left[\rho \varepsilon_{kmj} r_m f_j + (\varepsilon_{kmj} r_m t_{\ell j})_{,\ell} + m_{\ell k,\ell} + \rho \ell_k \right] dv . \tag{11}$$

Note that

$$(\varepsilon_{kmj} r_m t_{\ell j})_{,\ell} = \varepsilon_{kmj} t_{mj} + \varepsilon_{kmj} r_m t_{\ell j,\ell} . \tag{12}$$

Taking the cross product of $\mathbf{r}$ and equation (8), it follows that

$$\varepsilon_{kmj} r_m \dot{v}_j = \rho \varepsilon_{kmj} r_m f_j + \varepsilon_{kmj} r_m t_{\ell j,\ell} . \tag{13}$$

Using (12) and (13) in (11), we find

$$\rho \dot{\sigma}_k = \rho \ell_k + \varepsilon_{kmj} t_{mj} + m_{\ell k,\ell} . \quad \text{(Local)} \tag{14}$$

Equation (14) is the statement of local conservation of angular momentum for a polar media.

When

$$\sigma_k = \ell_k = m_{k\ell} = 0 , \tag{15}$$

Equation (14) reduces to

$$\varepsilon_{kmj} t_{mj} = 0 , \tag{16}$$

i.e., the stress tensor must be symmetric for a nonpolar media.

Axiom 4: Principle of Conservation of Energy

Time rate of change of internal and kinetic energy is equal to the rate of work done by the external force and the net heat transferred to the body. That is

$$\frac{d}{dt}(K + E) = W + Q . \tag{17}$$

Here, K is the kinetic energy, E is the internal energy, W is the rate of work done, and Q is the rate of heat transfer. Equation (17) may be restated as

$$\frac{d}{dt}\int_v \rho\left(e + \frac{1}{2} v_k v_k\right) dv = \int_v \rho v_k f_k dv + \int_s v_k \cdot t_k^{(n)} ds + \int_s q_k ds_k + \int_v \rho r dv , \tag{18}$$

Using the divergence theorem, we find

$$\int_v \rho(\dot{e} + v_k \dot{v}_k) dv = \int_v (\rho v_k f_k + v_k t_{\ell k,\ell} + t_{\ell k} v_{k,\ell} + q_{k,k} + \rho r) dv . \tag{19}$$

Taking the dot product of equation (8) with v_k and subtracting the result from (19), leads to the local form of the conservation of energy. That is

$$\rho\dot{e} = t_{\ell k} v_{\ell,k} + q_{k,k} + \rho r . \qquad \text{(Local)} \tag{20}$$

In these equations, e is the internal energy density, q_k is the heat flux vector pointing outward, and r is the internal heat source per unit mass.

Axiom 5: Entropy Inequality (Clausius-Duhem)

Time rate of change of the entropy minus the net heat transferred divided by the temperature must be positive. That is,

$$\frac{d}{dt}\int_v \rho\eta dv - \int_S \frac{q_k n_k}{T} ds - \int_v \frac{\rho r}{T} dv \geq 0 , \tag{21}$$

where η is the entropy density and T is the temperature.

Inequality (21) may be restated as

$$\int_v \left(\rho\dot{\eta} - (\frac{q_k}{T})_{,k} - \frac{\rho r}{T} \right) dv \geq 0 , \tag{22}$$

or

$$\rho\dot{\eta} - (\frac{q_k}{T})_{,k} - \frac{\rho r}{T} \geq 0 . \qquad \text{(Local)} \tag{23}$$

In summary, the basic conservation laws in vector notation are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{24}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{t} , \qquad \mathbf{t} = \mathbf{t}^T \tag{25}$$

$$\rho \frac{de}{dt} = \mathbf{t} : \nabla \mathbf{v} + \nabla \cdot \mathbf{q} + \rho r \tag{26}$$

$$\rho \frac{d\eta}{dt} - \nabla \cdot (\frac{\mathbf{q}}{T}) - \frac{\rho r}{T} \geq 0 \tag{27}$$

Continuum Thermodynamics and Constitutive Equations

Introducing the Helmholtz free energy function

$$\psi = e - T\eta \ , \tag{28}$$

entropy inequality (27) may be restated as

$$\frac{\rho}{T}(\dot{e} - \dot{T}\eta - \dot{\psi}) - \left(\frac{q_k}{T}\right)_{,k} - \frac{\rho r}{T} \ge 0 \,. \tag{29}$$

Here, we used

$$\dot{\psi} = \dot{e} - \dot{T}\eta - T\dot{\eta} \tag{30}$$

to eliminate $\dot{\eta}$.

Using the energy equation (26) to eliminate $\dot{e}$ in (29), we find

$$\frac{1}{T}\left[-\rho(\dot{\psi} + \eta\dot{T}) + t_{\ell k} v_{\ell,k} + \frac{q_k T_{,k}}{T} \right] \ge 0 \tag{31}$$

Inequality (31) is an alternative statement of the Clausius-Duhem inequality.

Constitutive Postulates

Assume that

$$\psi = \psi(T, \rho, d_{k\ell}, T_{,k}) \,, \tag{32}$$

where

$$d_{k\ell} = \frac{1}{2}(v_{k,\ell} + v_{\ell,k}) \tag{33}$$

is the deformation rate tensor. From (33), it follows that

$$\dot{\psi} = \frac{\partial \psi}{\partial T}\dot{T} + \frac{\partial \psi}{\partial \rho}\dot{\rho} + \frac{\partial \psi}{\partial d_{k\ell}}\dot{d}_{k\ell} + \frac{\partial \psi}{\partial T_{,k}}\dot{\overline{T_{,k}}} \,. \tag{34}$$

By definition, the thermodynamic pressure is given as

$$p = -\frac{\partial \psi}{\partial \rho^{-1}} = \rho^2 \frac{\partial \psi}{\partial \rho}. \tag{35}$$

Continuity equation implies that

$$\dot{\rho} = -\rho d_{kk}. \tag{36}$$

Thus,

$$\frac{\partial \psi}{\partial \rho}\dot{\rho} = -\frac{p}{\rho} d_{kk}. \tag{37}$$

Using (34) and (37) in (31) and collecting terms, we find

$$\frac{1}{T}\left[-\rho\left(\frac{\partial \psi}{\partial T} + \eta\right)\dot{T} + (t_{k\ell} + p\delta_{k\ell})d_{k\ell} - \rho\frac{\partial \psi}{\partial T_{,k}}\dot{\overline{T_{,k}}} - \rho\frac{\partial \psi}{\partial d_{k\ell}}\dot{d}_{k\ell} + \frac{q_k T_{,k}}{T}\right] \geq 0 \tag{38}$$

Inequality (38) must hold for all independent variations of $\dot{T}$, $\dot{\overline{T_{,k}}}$, $\dot{d}_{k\ell}$, $d_{k\ell}$, and $T_{,k}$. Thus, It follows that

$$\eta = -\frac{\partial \psi}{\partial T}, \tag{39}$$

$$\frac{\partial \psi}{\partial T_{,k}} = \frac{\partial \psi}{\partial d_{k\ell}} = 0, \tag{40}$$

and (38) reduces to

$$(t_{k\ell} + p\delta_{k\ell})d_{k\ell} + \frac{q_k T_{,k}}{T} \geq 0 \tag{41}$$

Linear Constitutive Equations

The general linear constitutive equations for the stress tensor and the heat flux vector are given as

$$t_{k\ell} = -p\delta_{k\ell} + L_{k\ell ij}d_{ij}, \tag{42}$$

$$q_k = L_{kj}T_{,j}, \tag{43}$$

Subjected to constraints,

$$L_{k\ell ij} d_{ij} d_{k\ell} \geq 0, \qquad L_{kj} T_{,k} T_{,j} \geq 0. \tag{44}$$

Here, $L_{k\ell ij}$ and L_{kj} are, respectively, a fourth order and a second order constant tensors.

For an isotropic fluid, $\mathbf{L}$'s must be isotropic tensors. The most general forms of isotropic tensors of fourth and second order are:

$$L_{k\ell ij} = \lambda\delta_{k\ell}\delta_{ij} + \mu(\delta_{ki}\delta_{\ell j} + \delta_{kj}\delta_{\ell i}) + \mu_1(\delta_{ki}\delta_{\ell j} - \delta_{kj}\delta_{\ell i}), \tag{45}$$

$$L_{k\ell} = \kappa\delta_{k\ell}, \tag{46}$$

where λ, μ, μ_1, and κ are the material constants that, in general, are functions of temperature.

Using (45) and (46) in (42) and (43) and noting that $d_{k\ell}$ is a symmetric tensor, we find

$$t_{k\ell} = (-p + \lambda d_{ii})\delta_{k\ell} + 2\mu d_{k\ell}, \tag{47}$$

$$q_k = \kappa T_{,k}. \tag{48}$$

These are Newton's laws of viscosity and Fourier's law of heat conductivity. Inequality (44) imposes the following constraints on the coefficients of viscosity and heat conductivity:

$$3\lambda + 2\mu \geq 0, \; \mu \geq 0, \; \kappa \geq 0. \tag{49}$$

Stokes assumed that

$$\lambda = -\frac{2}{3}\mu \tag{50}$$

so that the pressure is the negative of average normal stresses at t point. Stokes assumption given by Equation (50) leads to $p = -\frac{1}{3} t_{kk}$.

Navier-Stokes Equation

Using (47) in (25), we find

$$\rho \frac{dv_k}{dt} = -p_{,k} + \mu v_{k,jj} + (\lambda + \mu) v_{j,jk} + \rho f_k \,. \tag{51}$$

For an incompressible fluid,

$$\nabla \cdot \mathbf{v} = 0 \,, \tag{52}$$

and

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{f} \,. \tag{53}$$

Equations (52) and (53) are four equations for determining four unknowns $\mathbf{v}$, p for an incompressible flow.

Energy Equation

Using Equation (48) (for a non-constant κ) in (26), we find

$$\rho \frac{de}{dt} = \nabla \cdot (\kappa \nabla T) + t_{ij} v_{j,i} + \rho r \,. \tag{54}$$

Noting that

$$t_{ij} v_{j,i} = -p v_{k,k} + \Phi \,, \tag{55}$$

where the dissipation function is defined as

$$\Phi = \lambda v_{k,k} v_{i,i} + 2\mu d_{ij} v_{j,i} \,. \tag{56}$$

Noting that

$$p v_{k,k} = -\frac{p}{\rho} \frac{d\rho}{dt} = \rho \frac{d}{dt}\left(\frac{p}{\rho}\right) - \frac{dp}{dt} \tag{57}$$

and using (56) and (57) in Equation (54), we find

$$\rho \frac{dh}{dt} = \frac{dp}{dt} + \nabla \cdot (\kappa \nabla T) + \Phi + \rho r \,, \tag{58}$$

where

$$h = e + \frac{p}{\rho} \tag{59}$$

is the enthalpy.

For flows with constant properties, assuming the perfect gas relationship

$$dh = c_P dT, \; de = c_v dT, \tag{60}$$

where c_P and c_v are heat capacities at constant pressure and volume, we find

$$\rho c_P \frac{dT}{dt} = \frac{dP}{dt} + \kappa \nabla^2 T + \Phi + \rho r. \tag{61}$$

For incompressible fluids, the energy equation becomes

$$\rho c_v \frac{dT}{dt} = \kappa \nabla^2 T + \Phi + \rho r, \tag{62}$$

where

$$\Phi = \mu (v_{i,j} + v_{j,i}) v_{j,i} \tag{63}$$

Density Change Due to Temperature Variation

For incompressible fluids, Boussinesq assumed that

$$\rho = \rho_0 (1 - \beta (T - T_0)), \; \beta = \text{const.} \tag{64}$$

When the body flow is only due to gravity, we have

$$\rho \mathbf{f} = -\rho_0 g \mathbf{k} [1 - \beta (T - T_0)]. \tag{65}$$

Using (65) in (53), we find

$$\rho_0 \frac{d\mathbf{v}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{v} - \rho_0 g [1 - \beta (T - T_0)] \mathbf{k}. \tag{66}$$

or

$$\rho_0 \frac{d\mathbf{v}}{dt} = -\nabla \hat{P} + \mu \nabla^2 \mathbf{v} - \rho_0 g \beta (T - T_0) \mathbf{k}, \tag{67}$$

where we have defined excess pressure beyond hydrostatics as

$$\hat{P} = p + \rho_0 g z \,. \tag{68}$$

For a general body force, we find

$$\rho_0 \frac{d\mathbf{v}}{dt} = -\nabla\hat{P} + \mu\nabla^2\mathbf{v} - \rho_0\beta(T - T_0)\mathbf{f} \,. \tag{69}$$

Dimensionless Equations

It is advantageous to express the governing equations in nondimensional forms. We introduce dimensionless quantities:

$$x_i^* = \frac{x_i}{L}, \ \mathbf{v}^* = \frac{\mathbf{v}}{U_\infty}, \ t^* = \frac{tU_\infty}{L}, \ \rho^* = \frac{\rho}{\rho_0}, \tag{70}$$

$$P^* = \frac{\hat{P} - P_\infty}{\rho_0 U_\infty^2}, \ T^* = \frac{T - T_0}{\Delta T_0}, \ \mathbf{f}^* = \frac{\mathbf{f}}{g} \tag{71}$$

where L, U_∞, ρ_0 and T_0 are length, velocity, density and temperature scales. Using (70), the equations of motion and energy transport in nondimensional form become

$$\frac{\partial\rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \mathbf{v}^*) = 0 \,, \tag{72}$$

$$\rho^* \frac{d\mathbf{v}^*}{dt^*} = -\nabla^* P^* + \frac{1}{Re}\nabla^{*2}\mathbf{v}^* - \frac{Gr}{Re^2} T^* \mathbf{f}^* \,, \tag{73}$$

$$\rho^* \frac{dT^*}{dt^*} = Ec\frac{dP^*}{dt^*} + \frac{1}{Re\,Pr}\nabla^{*2}T^* + \frac{Ec}{Re}\Phi^* \,. \tag{74}$$

Here, we have defined the following dimensionless groups:

$$\text{Reynolds number} = Re = \frac{\rho_0 U_\infty L}{\mu}, \tag{75}$$

$$\text{Prandtl number} = Pr = \frac{\mu c_P}{\kappa}, \tag{76}$$

Eckert number: $Ec = \frac{U_\infty^2}{c_P \Delta T_0}$, (77)

Grashof number: $Gr = \frac{g\beta\rho_0^2 L^3 \Delta T_0}{\mu^2}$ (78)