CONTINUUM FLUID MECHANICS

Motion

A body is a collection of material particles. The point $\mathbf{X}$ is a material point and it is the position of the material particles at time zero.

Definition: A one-to-one one-parameter mapping

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$

is called motion. The inverse

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$$

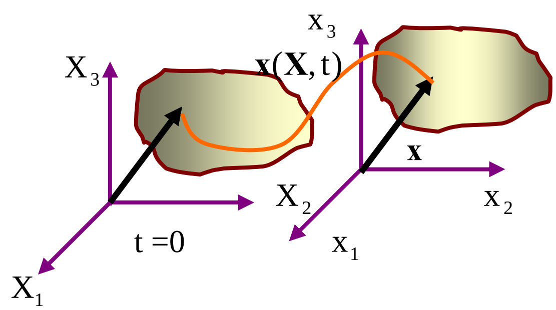

Figure 1. Schematic of motion.

is the inverse motion. X_k are referred to as the material coordinates of particle $\mathbf{x}$, and $\mathbf{x}$ is the spatial point occupied by $\mathbf{X}$ at time t.

Theorem: The inverse motion exists if the Jacobian, J of the transformation is nonzero. That is

$$J = \det\left|\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right| = \det\left|\frac{\partial x_k}{\partial X_K}\right| \neq 0 .$$

This is the statement of the fundamental theorem of calculus.

Definition: Streamlines are the family of curves tangent to the velocity vector field at time t.

Given a velocity vector field $\mathbf{v}$, streamlines are governed by the following equations:

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3} \qquad \text{for a given t.}$$

Definition: The streak line of point $\mathbf{x}^0$ at time t is a line, which is made up of material points, which have passed through point $\mathbf{x}^0$ at different times $\tau \leq t$.

Given a motion $x_i = x_i(\mathbf{X}, t)$ and its inverse $X_K = X_K(\mathbf{x}, t)$, it follows that the material particle X_k^0 will pass through the spatial point $\mathbf{x}^0$ at time τ. i.e.,

$X_k^0 = X_k^0(\mathbf{x}^0, \tau)$.

Thus, the equation for the streak line of $\mathbf{x}^0$ at time t is given by

$$x_i = x_i(\mathbf{X}^0(\mathbf{x}^0, \tau), t) \qquad \text{for fixed } t.$$

Deformation Rate Tensors

Suppose a motion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is given.

Definition: Deformation Gradient

$$dx_k = \frac{\partial x_k}{\partial x_K} dx_K = x_{k,K} dX_K . \tag{1}$$

$x_{k,K} = \frac{\partial x_k}{\partial X_K}$ is referred to as the deformation gradient tensor.

$X_{K,k} = \frac{\partial X_K}{\partial x_k}$ is the (inverse) deformation gradient tensor.

Definition: Deformation Tensors

An element of arc in the deformed body is given as

$$ds^2 = dx_k dx_\ell \delta_{k\ell} . \tag{2}$$

The distance from the origin in the un-deformed body is given by

$$dS^2 = dX_K X_L \delta_{KL} \tag{3}$$

Using the deformation gradient, (1) may be restated as

$$ds^2 = x_{k,K} dX_K x_{\ell,L} dX_L \delta_{kl} = C_{KL} dX_K X_L , \tag{4}$$

where $C_{KL} = \delta_{kl} x_{k,K} x_{\ell,L}$ is the Green deformation tensor.

Similarly, (3) may be rewritten as

$$dS^2 = X_{K,k} X_{L,\ell} \delta_{KL} dx_k dx_\ell = c_{k\ell} dx_k dx_\ell , \tag{5}$$

where $c_{kl} = X_{K,k} X_{L,\ell} \delta_{KL}$ is the Cauchy deformation tensor.

Definition: Strain Tensors

The change in the square of arc length during the deformation is given by

$$ds^2 - dS^2 = (C_{KL} - \delta_{KL})dX_K dX_L = 2E_{KL} dX_K dX_L \,, \tag{6}$$

Here, we introduced the Lagrangian strain tensor

$$2E_{KL} = C_{KL} - \delta_{KL} \,. \tag{7}$$

Similarly, (6) may be restated as

$$ds^2 - dS^2 = (\delta_{kl} - c_{kl})dx_k dx_\ell = 2e_{k\ell} dx_k dx_\ell \,, \tag{8}$$

where the Eulerian strain tensor is defined as

$$2e_{k\ell} = \delta_{k\ell} - c_{k\ell} \,. \tag{9}$$

Partial and Total Time Derivatives

Let A be any scalar or tensor quantity. The partial time derivative is defined as

$$\frac{\partial A}{\partial t} = \left.\frac{\partial A}{\partial t}\right|_{\mathbf{x}} . \tag{10}$$

The material derivative (total time derivative) is defined as

$$\frac{dA}{dt} = \left.\frac{\partial A}{\partial t}\right|_{\mathbf{X}} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x_i}\left.\frac{\partial x_i}{\partial t}\right|_{\mathbf{X}} . \tag{11}$$

Definition: Velocity

$$v_i = \left.\frac{\partial x_i}{\partial t}\right|_{\mathbf{X}} = \frac{dx_i}{dt} = \dot{x}_i \,. \tag{12}$$

Definition: Acceleration

$$a_i = \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} . \tag{13}$$

Definition: Path lines

The curve in space along which the material point $\mathbf{x}$ travels is referred to as the path line of the material particle $\mathbf{X}$.

The equation for the path line of $\mathbf{x}$ is

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \quad \text{for fixed } \mathbf{X}. \tag{14}$$

If the velocity field is known, then equations

$$\frac{dx_i}{dt} = v_i(\mathbf{x}, t) \quad \text{for } i = 1,2,3 \tag{15}$$

must be solved for evaluating the path lines.

Deformation Rate Tensor

Material derivatives of dx_k and deformation gradients are given as

$$\frac{d}{dt}(dx_k) = \frac{d}{dt}(x_{k,K} dX_K) = v_{k,K} dX_K = v_{k,\ell} dx_\ell \tag{16}$$

$$\frac{d}{dt}(x_{k,K}) = \frac{\partial}{\partial X_K}\frac{dx_k}{dt} = v_{k,K} = v_{k,\ell} x_{\ell,K} \tag{17}$$

Theorem: The material derivative of the square of the arc length is given by

$$\frac{d}{dt}(ds^2) = 2 d_{k\ell} dx_k dx_\ell \tag{18}$$

where $d_{k\ell} = \frac{1}{2}(v_{k,\ell} + v_{\ell,k})$ is the Eulerian deformation rate tensor.

Proof:

$$\frac{d}{dt}(ds^2) = \frac{d}{dt}(\delta_{k\ell} dx_k dx_\ell) = \delta_{k\ell}\left(\frac{dx_k}{dt} dx_\ell + dx_k \frac{dx_\ell}{dt}\right)$$

Now using (16), we find

$$\frac{d}{dt}(ds^2) = \delta_{k\ell}\left(v_{k,m}dx_m dx_\ell + v_{\ell,m}dx_m dx_k\right) = v_{k,m}dx_m dx_k + v_{\ell,m}dx_m dx_\ell$$
$$= (v_{k,\ell} + v_{\ell,k})dx_k dx_\ell = 2d_{k\ell}dx_k dx_\ell$$

Relationships between Deformation Rate Tensors and Deformation Strain Tensors

From (4), recall that $ds^2 = C_{KL}dX_K dX_L$. Taking the material derivative, we find

$$\frac{d}{dt}(ds^2) = \dot{C}_{KL}dX_K dX_L = \dot{C}_{KL}X_{K,k}X_{L,\ell}dx_k dx_\ell \tag{19}$$

Equation (18) implies that

$$\frac{d}{dt}(ds^2) = 2d_{k\ell}dx_k dx_\ell = 2d_{k\ell}dx_{k,K}x_{\ell,L}dX_K dX_L \tag{20}$$

From (7), it follows that

$$\dot{C}_{KL} = 2\dot{E}_{KL}. \tag{21}$$

From (19)-(21), we find

$$\dot{C}_{KL} = 2\dot{E}_{KL} = 2d_{k\ell}x_{k,K}x_{\ell,L} \tag{22}$$

and

$$2d_{k\ell} = \dot{C}_{KL}X_{K,k}X_{L,\ell} = 2\dot{E}_{KL}X_{K,k}X_{L,\ell} \tag{23}$$

Equations (22) and (23) show the relationship between the deformation rate tensor $d_{k\ell}$ and the material derivative of Green's deformation tensor and Lagrangian strain tensor.

Rivlin-Ericksen Tensors

Definition: The Rivlin-Ericksen tensor of order n is defined as

$$\frac{d^n}{dt^n}(ds^2) = A^{(n)}_{k\ell}dx_k dx_\ell. \tag{24}$$

Clearly,

$$A^{(1)}_{k\ell} = 2d_{k\ell}. \tag{25}$$

The Rivlin-Ericksen tensor satisfies the following recurrence relationship:

$$A_{k\ell}^{(n+1)} = \frac{d}{dt} A_{k\ell}^{(n)} + A_{km}^{(n)} v_{m,\ell} + A_{\ell m}^{(n)} v_{m,k} \qquad (26)$$

For example,

$$A_{k\ell}^{(2)} = 2\dot{d}_{k\ell} + 2d_{km} v_{m,\ell} + 2d_{\ell m} v_{m,k} \qquad (27)$$

The Rivlin-Ericksen tensors are important tensors for certain viscoelastic materials.

Lemma: The material derivative of the Jacobian is given by

$$\frac{d}{dt} J = J v_{k,k} \, . \qquad (28)$$

Time Rate of Change of Volume Element

Noting that

$$dv = J dV \, , \qquad (29)$$

It follows that

$$\frac{d}{dt} dv = \frac{dJ}{dt} dV = v_{k,k} J dV \, ,$$

or

$$\frac{d}{dt} dv = v_{k,k} dv \, . \qquad (30)$$

Reynolds Transport Theorem

The material derivative of an integral taken over a material volume is given as

$$\frac{d}{dt} \iiint_v f dv = \iiint_v \frac{\partial f}{\partial t} dv + \oint\!\!\oint_s f \mathbf{v} \cdot \mathbf{ds} \, . \qquad (31)$$

Proof:

$$\frac{d}{dt} \iiint_v f dv = \frac{d}{dt} \iiint_V f \, J dV = \iiint_V \left(\frac{df}{dt} J + f \, \frac{dJ}{dt} \right) dV$$

Using (30), we find

$$\frac{d}{dt}\iiint_v f dv = \iiint_V \left(\frac{df}{dt} + v_{k,k} f\right) J dV = \iiint_v (\dot{f} + v_{k,k} f) dv$$

Noting that $\dot{f} = \frac{\partial f}{\partial t} + v_k \frac{\partial f}{\partial x_k}$, we find

$$\frac{d}{dt}\iiint_v f dv = \iiint_v \left(\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_k}(v_k f)\right) dv$$

Using the divergence theorem (25) follows.

Spin and Vorticity

Spin tensor is defined as:

$$\omega_{k\ell} = \frac{1}{2}(v_{k,\ell} - v_{\ell,k}) \qquad (32)$$

Vorticity vector is defined as:

$$\zeta_i = \varepsilon_{ijk}\omega_{kj} = \varepsilon_{ijk} v_{k,j} \qquad (33)$$

Angular velocity vector is defined as

$$\omega_i = \frac{1}{2}\zeta_i . \qquad (34)$$

In vector notation, we have

$$\boldsymbol{\zeta} = \nabla \times \mathbf{v}, \ \boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v} \qquad (35)$$

Note that

$$(\nabla \mathbf{v})^T = \mathbf{d} + \boldsymbol{\omega}, \ (v_{i,j} = d_{ij} + \omega_{ij}) \qquad (36)$$