ME 639 - Turbulence

Clarkson University

Continuum Mechanics -

Kinematics

Goodarz Ahmadi

Department of Mechanical and Aeronautical Engineering
Clarkson University
Potsdam, NY 13699-5725

Kinematics

Clarkson University

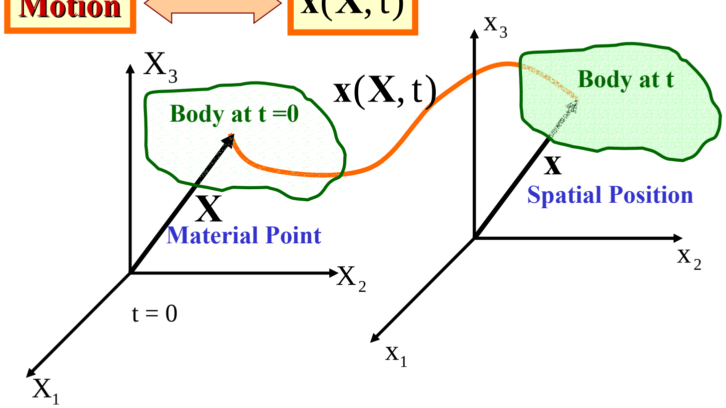
Outline

- Motion & Inverse Motion
- Time Derivatives
- Velocity and Acceleration
- Deformation Rate Tensor
- Spin Tensor & Vorticity

Kinematics

Clarkson University

Motion ⟷ $\mathbf{x}(\mathbf{X}, t)$



Motion

Clarkson University

Body = Collection of Material Particles

X= Material Point = Position of particle at time zero

Motion: $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$

Inverse Motion: $\mathbf{X}=\mathbf{X}(\mathbf{x},t)$

Jacobian

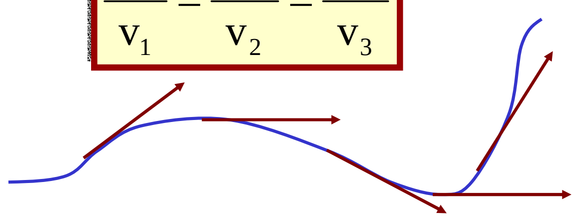
$$J = \det\left|\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right| = \det\left|\frac{\partial x_k}{\partial X_K}\right| \neq 0$$

Streamlines

Clarkson University

Streamlines are curves tangent to the velocity vector field

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}$$



Streak Lines

Clarkson University

The streak line of point x^o at time t is a line, which is made up of material points, that have passed through point x^o at different times $\tau \le t$

X_k^0 passes through $\mathbf{x}^0$ at time τ

$$X_k^0 = X_k^0(\mathbf{x}^0, \tau)$$

Streak lines

$$x_i = x_i(\mathbf{X}^0(\mathbf{x}^0, \tau), t)$$

For fixed t

Deformation Gradient

Clarkson University

$$dx_k = \frac{\partial x_k}{\partial X_K} dX_K = x_{k,K} dX_K$$

$$x_{k,K} = \frac{\partial x_k}{\partial X_K}$$

Deformation Gradient

$$X_{K,k} = \frac{\partial X_K}{\partial x_k}$$

Inverse Deformation Gradient

Deformation Tensors

Clarkson University

Element of Arc in the Deformed Body

$$ds^2 = dx_k dx_\ell \delta_{k\ell}$$

Element of Arc in the Undeformed Body

$$dS^2 = dX_K X_L \delta_{KL}$$

Green Deformation Tensor

$$C_{KL} = \delta_{kl} x_{k,K} x_{\ell,L}$$

$$ds^2 = C_{KL} dX_K X_L$$

Cauchy Deformation Tensor

$$c_{kl} = X_{K,k} X_{L,\ell} \delta_{KL}$$

$$dS^2 = c_{k\ell} dx_k dx_\ell$$

Strain Tensors

Clarkson University

Lagrangian Strain Tensor → $$2E_{KL} = C_{KL} - \delta_{KL}$$

$$ds^2 - dS^2 = 2E_{KL}dX_K dX_L$$

Eulerian Strain Tensor → $$2e_{k\ell} = \delta_{k\ell} - c_{k\ell}$$

$$ds^2 - dS^2 = 2e_{k\ell}dx_k dx_\ell$$

Partial and Total Time Derivatives

Clarkson University

Partial Time Derivatives → $$\frac{\partial A}{\partial t} = \left.\frac{\partial A}{\partial t}\right|_x$$

Material Time Derivatives → $$\frac{dA}{dt} = \left.\frac{\partial A}{\partial t}\right|_X = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x_i}\left.\frac{\partial x_i}{\partial t}\right|_X$$

Velocity → $$v_i = \left.\frac{\partial x_i}{\partial t}\right|_X = \frac{dx_i}{dt} = \dot{x}_i$$

Acceleration → $$a_i = \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}$$

Path Line

Clarkson University

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$$ → **For Fixed X**

$$\frac{dx_i}{dt} = v_i(\mathbf{x}, t)$$ → **Path Line**

Time Derivatives

$$\frac{d}{dt}(dx_k) = \frac{d}{dt}(x_{k,K}dX_K) = v_{k,K}dX_K = v_{k,\ell}dx_\ell$$

$$\frac{d}{dt}(x_{k,K}) = \frac{\partial}{\partial X_K}\frac{dx_k}{dt} = v_{k,K} = v_{k,\ell}x_{\ell,K}$$

Deformation Rate Tensor

Clarkson University

$$d_{k\ell} = \frac{1}{2}(v_{k,\ell} + v_{\ell,k})$$

$$\frac{d}{dt}(ds^2) = 2d_{k\ell}dx_k dx_\ell$$

Identities

$$\dot{C}_{KL} = 2\dot{E}_{KL} = 2d_{k\ell}x_{k,K}x_{\ell,L}$$

$$2d_{k\ell} = \dot{C}_{KL}X_{K,k}X_{L,\ell} = 2\dot{E}_{KL}X_{K,k}X_{L,\ell}$$

Rivlin-Ericksen Tensors

Clarkson University

$$A^{(n+1)}_{k\ell} = \frac{d}{dt} A^{(n)}_{k\ell} + A^{(n)}_{km} v_{m,\ell} + A^{(n)}_{\ell m} v_{m,k}$$

$$A^{(1)}_{k\ell} = 2 d_{k\ell}$$

$$A^{(2)}_{k\ell} = 2\dot{d}_{k\ell} + 2 d_{km} v_{m,\ell} + 2 d_{\ell m} v_{m,k}$$

$$\frac{d^n}{dt^n}(ds^2) = A^{(n)}_{k\ell} dx_k dx_\ell$$

Rate of Volume Change

Clarkson University

$$\frac{d}{dt} J = J v_{k,k}$$

$$dv = J dV$$

$$\frac{d}{dt} dv = v_{k,k} dv$$

Reynolds Transport Theorem

Clarkson University

$$\frac{d}{dt}\iiint_V f dv = \iiint_V \frac{\partial f}{\partial t} dv + \oiint_S f \mathbf{v} \cdot \mathbf{ds}$$

Proof

$$\frac{d}{dt}\iiint_v f dv = \frac{d}{dt}\iiint_V f\, J dV = \iiint_V (\frac{df}{dt} J + f \frac{dJ}{dt}) dV$$

$$\frac{d}{dt}\iiint_v f dv = \iiint_V (\frac{df}{dt} + v_{k,k} f) J dV = \iiint_v (\dot{f} + v_{k,k} f) dv$$

$$\frac{d}{dt}\iiint_v f dv = \iiint_v \left(\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_k}(v_k f) \right) dv$$

Spin and Vorticity

Clarkson University

Spin Tensors → $\omega_{k\ell} = \frac{1}{2}(v_{k,\ell} - v_{\ell,k})$

Vorticity Vector $\zeta_i = \varepsilon_{ijk}\omega_{kj} = \varepsilon_{ijk} v_{k,j}$

Angular Velocity Vector $\omega_i = \frac{1}{2}\zeta_i$ $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v}$

$$(\nabla \mathbf{v})^T = \mathbf{d} + \boldsymbol{\omega}$$

$$\zeta = \nabla \times \mathbf{v}$$