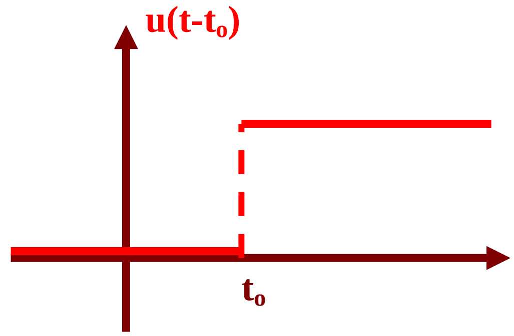
Review of Engineering Mathematics

Special Functions

Unit step function

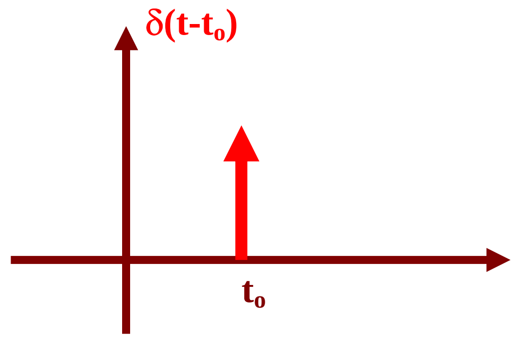
$$u(t-t_0)=\begin{cases}1 & t\ge t_0\\ 0 & t<t_0\end{cases}$$



Delta Function

Dirac delta function is defined as

$$\delta(t-t_0)=\frac{du(t-t_0)}{dt}$$



Note that

$$\int_{-\infty}^{+\infty}\delta(t-t_0)dt=\int_{t_0-\varepsilon}^{t_0+\varepsilon}\delta(t-t_0)dt=1$$

Also

$$\int_{-\infty}^{+\infty}f(t)\delta(t-t_0)dt=\int_{t_0-\varepsilon}^{t_0+\varepsilon}f(t)\delta(t-t_0)dt=f(t_0)$$

$$\int_{-\infty}^{t}f(t_1)\delta(t_1-t_0)dt_1=f(t_0)u(t-t_0)$$

$$\delta[a(t-t_0)] = \frac{1}{|a|}\delta(t-t_0) \qquad \text{for} \qquad a \neq 0$$

Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\,,\ \operatorname{erfc}(x) = 1-\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

$$\operatorname{erf}(0) = 0 = \operatorname{erfc}(\infty), \quad \operatorname{erf}(-x) = -\operatorname{erf}(x)$$

Fourier Transform

Define Fourier Transform (Exponential)

$$\bar{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x'} f(x')dx'$$

The inverse transform is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega x}\bar{f}(\omega)d\omega$$

The above two equations are a Fourier Exponential Transform Pair.

$$\overline{f_c}(\omega) = \int_0^\infty \cos\omega x' f(x')dx'$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\,\overline{f_c}(\omega)d\omega$$

Applications to Differential Equations

Fourier Exponential Transform of derivatives

$$\Im\left\{\frac{df}{dx}\right\} = \int_{-\infty}^{+\infty} e^{-i\omega x}\frac{df(x)}{dx}dx = i\omega\bar{f}(\omega)$$

$$\Im\left\{\frac{d^2f}{dx^2}\right\} = -\omega^2\bar{f}(\omega), \quad \Im\left\{\frac{d^nf}{dx^n}\right\} = (i\omega)^n\bar{f}(\omega)$$

Example: Find f that satisfies the following differential equation:

$$\frac{d^2f}{dx^2} + a\frac{df}{dx} + bf = \delta(x - x_0) \qquad -\infty < x < +\infty$$

Take Fourier Exponential Transform

$$-\omega^2 \bar{f}(\omega) + ai\omega\bar{f}(\omega) + b\bar{f}(\omega) = e^{-i\omega x_0}$$

$$\bar{f}(\omega) = \frac{e^{-i\omega x_0}}{b - \omega^2 + ia\omega}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\omega(x - x_0)}}{b - \omega^2 + ia\omega} d\omega$$

Table of Fourier Exponential Transform Pair

$$\bar{f}(\omega)=\int_{-\infty}^{+\infty} e^{-i\omega x} f(x)dx, \qquad f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega x}\bar{f}(\omega)d\omega$$

$f(x)$	$\bar{f}(\omega)$
$f_1(x)e^{-i\omega_0 x}$	$\bar{f}_1(\omega+\omega_0)$
$f_1(x+x_0)$	$e^{i\omega x_0}\bar{f}(\omega)$
$f_1(x)*f_2(x)=\int_{-\infty}^{+\infty} f_1(\xi)f_2(x-\xi)d\xi$	$\bar{f}_1(\omega)\bar{f}_2(\omega)$
$\delta(x-x_0)$	$e^{-i\omega x_0}$
$e^{i\omega_0 x}$	$2\pi\delta(\omega-\omega_0)$
$e^{-\alpha\|x\|}$	$\frac{2\alpha}{\omega^2+\alpha^2}$
$\cos\omega_0 x$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$e^{-\alpha\|x\|}\cos\beta x$	$\frac{2\alpha(\omega^2+\alpha^2+\beta^2)}{(\omega^2-\beta^2-\alpha^2)^2+4\alpha^2\omega^2}$
$e^{-\alpha\|x\|}\left[\cos\beta x+\frac{\alpha}{\beta}\sin\beta\|x\|\right]$	$\frac{4\alpha(\alpha^2+\beta^2)}{(\omega^2-\beta^2-\alpha^2)^2+4\alpha^2\omega^2}$
$e^{-\alpha^2x^2}\cos\beta x$	$\frac{\sqrt{\pi}}{2\alpha}\left[\exp\left\{-\frac{(\omega+\beta)^2}{4\alpha^2}\right\}+\exp\left\{-\frac{(\omega-\beta)^2}{4\alpha^2}\right\}\right]$
$e^{-\alpha^2x^2}$	$\frac{\sqrt{\pi}}{\alpha}\exp\left\{-\frac{\omega^2}{4\alpha^2}\right\}$
$\frac{d^n}{dx^n}\delta(x)$	$(i\omega)^n$
$J_0(x)$	$\begin{cases}\frac{2}{\sqrt{1-\omega^2}} & \|\omega\|<1\\ 0 & \text{elsewhere}\end{cases}$

Probability and Random Processes

In this section, capital letters identifies a random variable and lower case letters are used for coordinate systems.

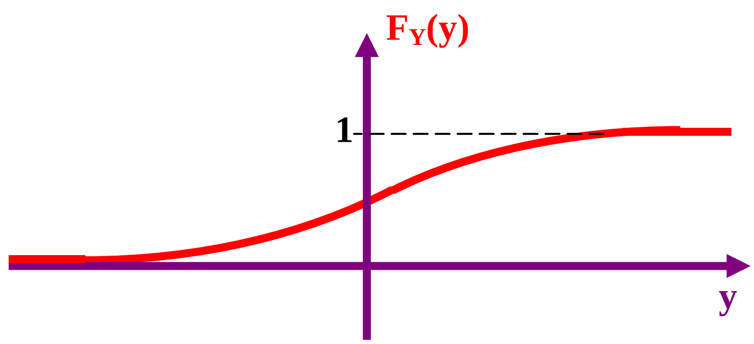
Distribution Function

The distribution function of a random variable Y is defined as the probability that $\{Y \le y\}$. That is

$$F_Y(y) = P\{Y \le y\}$$

It then follows that $F_Y(y)$ is monotonically increasing function and

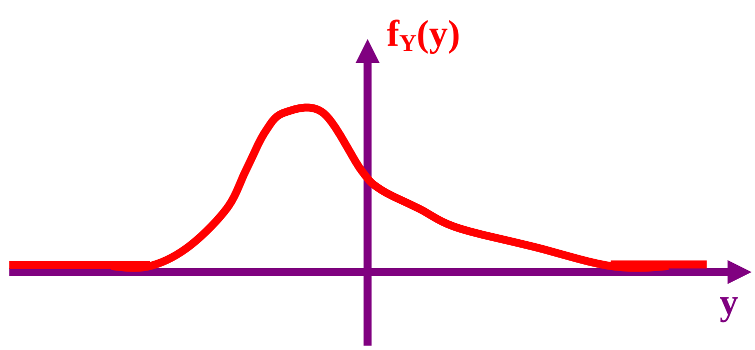
$$0 \le F_Y(y) \le 1.$$



Density Function

The probability density function is defined as

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$



Properties:

$$F_Y(\infty) = 1 = \int_{-\infty}^{+\infty} f_Y(y)dy, \qquad P\{y_1 < Y \le y_2\} = \int_{y_1}^{y_2} f_Y(y)dy = F_Y(y_2) - F_Y(y_1)$$

Expected Value

$$E\{Y\} = \overline{Y} = \int_{-\infty}^{+\infty} y f_Y(y)dy$$

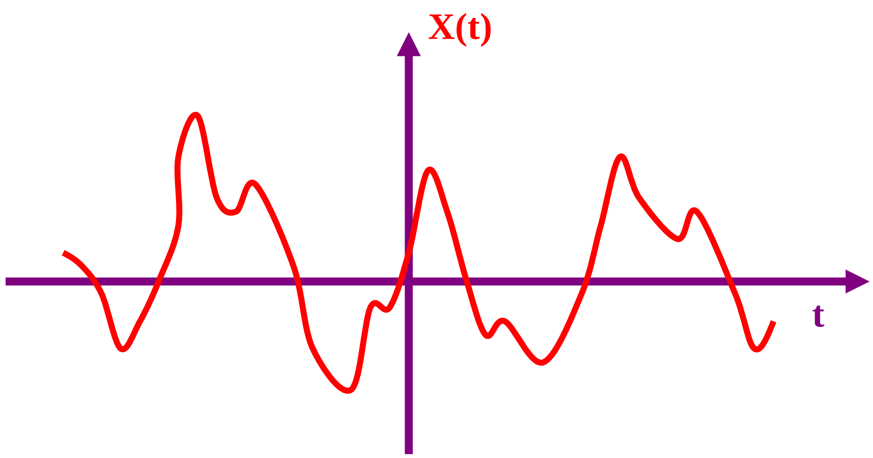
$$E\{g(Y)\} = \overline{g(Y)} = \int_{-\infty}^{+\infty} g(y) f_Y(y)dy$$

Variance

$$\sigma_Y^2 = E\{(Y-\overline{Y})^2\} = E\{Y^2\} - \overline{Y}^2$$

Stochastic Process

Ensembles of random functions of time (or space) are referred to as stochastic processes. For fixed time, a stochastic process becomes a random variable. Every sample of a stochastic process is a time function.



Statistics of a stochastic process may be evaluated similar to those of a random variable. For example, the mean value is given as

$$E\{X(t)\} = \int_{-\infty}^{+\infty} x f_X(x,t)\,dx$$

Time Average

Time averaging over an interval (0,T) is defined as

$$\overline{X}(t) = \frac{1}{T}\int_0^T X(t)\,dt \approx E\{X(t)\}$$

Autocorrelation

The autocorrelation of a random process is defined as

$$R_{xx}(\tau) = E\{X(t+\tau)X(t)\} = \frac{1}{T}\int_0^T X(t+\tau)X(t)\,dt$$

where τ is the time difference, an it is assumed that X(t) is a stationary random process. Note that

$$R_{xx}(0) = E\{X^2(t)\} = \overline{X^2(t)}$$

Energy Spectrum

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau} R_{xx}(\tau) d\tau$$

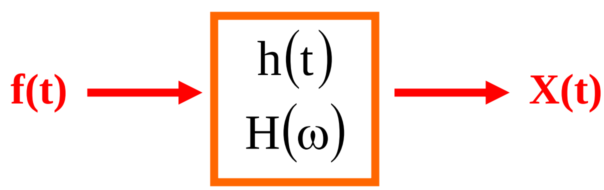
$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega\tau} S_{xx}(\omega) d\omega$$

It may be shown that

$$S_{xx}(\omega) = \frac{1}{T} \left| \tilde{X}(\omega) \right|^2, \qquad \tilde{X}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} X(t) dt$$

Linear Systems

Consider a linear system with impulse response h(t) and a system function $H(\omega)$ for $X(0) = 0$ as shown schematically in the figure.



The solution then is given as

$$X(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

where

More generally,

$$X(t) = \int_{-\infty}^{+\infty} h(t-\tau) f(\tau) d\tau = h(t) * f(t)$$

Taking Fourier Transform $\quad H(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} h(t) dt$

$$\tilde{x}(\omega) = H(\omega)\tilde{f}(\omega)$$

$$S_{xx}(\omega) = \frac{1}{T}\left|\bar{x}(\omega)\right|^2 = \frac{1}{T}\left|H(\omega)\right|^2\left|\bar{f}(\omega)\right|^2 = \left|H(\omega)\right|^2 S_{ff}(\omega)$$

$$S_{xx}(\omega) = \left|H(\omega)\right|^2 S_{ff}(\omega)$$

For

$$\dot{x} + \alpha x = f(t),$$

$$h(t) = e^{-\alpha t}$$

For

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = f(t),$$

$$h(t) = \frac{1}{\omega_d} e^{-\zeta\omega_0 t}\sin\omega_d t \qquad \omega_d = \omega_0\sqrt{1-\zeta^2}$$

Useful Integrals

$$\int e^{ax} P(x) dx = \frac{e^{ax}}{a}\left[P(x) - \frac{1}{a} P'(x) + \frac{1}{a^2} P''(x) - \frac{1}{a^3} P'''(x) \ldots \right]$$

$$\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$$

$$\int x \sin ax dx = \frac{\sin ax - ax \cos ax}{a^2}$$

$$\int x \cos ax dx = \frac{\cos ax + ax \sin ax}{a^2}$$

$$\int \ln ax dx = x \ln ax - x$$

$$\int x \ln ax dx = \frac{x^2}{2} \ln ax - \frac{x^2}{4}$$

$$\int x \sinh ax dx = \frac{ax \cosh ax - \sinh ax}{a^2}$$

$$\int x \cosh ax dx = \frac{ax \sinh ax - \cosh ax}{a^2}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right), \qquad \int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) = \frac{1}{2a} \ln \frac{x - a}{x + a}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int e^{ax} x dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

Vector Identities

$$\nabla \cdot \nabla \times \vec{u} = 0 \qquad \nabla \times (\nabla \varphi) = 0$$

$$\nabla \times \nabla \times \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

$$\vec{u} \cdot \nabla \vec{u} = \nabla\left(\frac{u^2}{2}\right) - \vec{u} \times (\nabla \times \vec{u})$$

$$\nabla \times (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \vec{u} - \vec{u} \cdot \nabla \vec{v} + (\nabla \cdot \vec{v})\vec{u} - (\nabla \cdot \vec{u})\vec{v}$$

$$\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \times \vec{u} - \vec{u} \cdot \nabla \times \vec{v}$$

$$\nabla(\vec{u} \cdot \vec{v}) = \vec{v} \cdot \nabla \vec{u} + \vec{u} \cdot \nabla \vec{v} + \vec{v} \times (\nabla \times \vec{u}) + \vec{u} \times (\nabla \times \vec{v})$$

Stokes Theorem

$$\oint_c \vec{u} \cdot d\vec{c} = \int_s (\nabla \times \vec{u}) \cdot ds$$

Divergence Theorem

$$\int_V \nabla \cdot \vec{u} dV = \int_S \vec{u} \cdot d\vec{s}$$